A Notation for Substitutions and Related Rates

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1 Motivation: Problems with Leibniz Notation

1.1 Related Rates - First Derivative

The canonical example of Leibniz notation used for related rates is as follows. Find the derivative of y with respect to x:

$$x = f(t)$$

$$y = g(t)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{g'(t)}{f'(t)}$$

It is not clear how one could arrive at this result without Leibniz notation.

1.2 Related Rates - Second Derivative

However, if one is not careful, Leibniz notation can produce erroneous results for second derivatives and higher:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{(\mathrm{d}t)^2}\frac{(\mathrm{d}t)^2}{(\mathrm{d}x)^2} = \frac{g''(t)}{f'(t)^2}$$

Which is incorrect. The correct derivation is as follows:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{g'(t)}{f'(t)} = \left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)\left(\frac{f'(t)g''(t) - g'(t)f''(t)}{f'(t)^2}\right)$$

$$= \frac{f'(t)g''(t) - g'(t)f''(t)}{f'(t)^3}$$

It is possible to change Leibniz notation for higher derivatives so that it is valid to manipulate it in the straight-forward manner, however, the changed notation is extremely unwieldly.[1]

[1] See "Extending the Algebraic Manipulability of Differentials" by Jonathan Bartlett and Asatur Khurshudyan, `https://arxiv.org/abs/1801.09553`

1.3 Differential Equation Substitution

And even so, there are still ways to make mistakes even when not using the higher derivative notation explicitly. Consider the following problem:

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} = f(y) \tag{1}$$

We will make the substitution

$$v = \frac{dy}{dt} \tag{2}$$

to reduce the order of the equation, rewriting it as an equation of v in terms of y. So we need to rewrite $\frac{\mathrm{d}^3 y}{\mathrm{d}t^3}$ in terms of v and y. We will start by rewriting $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}$.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}y} = v\frac{\mathrm{d}v}{\mathrm{d}y} \tag{3}$$

Now

$$\frac{\mathrm{d}^3 y}{\mathrm{d}t^3} = \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v\frac{\mathrm{d}v}{\mathrm{d}y}\right) \tag{4}$$

$$= \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}y} + v\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}y} \tag{5}$$

Using (3):

$$= \left(v\frac{\mathrm{d}v}{\mathrm{d}y}\right)\frac{\mathrm{d}v}{\mathrm{d}y} + v\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}y}v \tag{6}$$

$$= v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}}{\mathrm{d}t}v \tag{7}$$

$$= v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}v}{\mathrm{d}t} \tag{8}$$

Using (3):

$$= v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v\frac{\mathrm{d}}{\mathrm{d}y}\left(v\frac{\mathrm{d}v}{\mathrm{d}y}\right) \tag{9}$$

$$= v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v\left(\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v\frac{\mathrm{d}^2 v}{\mathrm{d}y^2}\right) \tag{10}$$

$$= 2v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v^2\frac{\mathrm{d}^2 v}{\mathrm{d}y^2} \tag{11}$$

This result is incorrect. The correct expression is

$$v\left(\frac{\mathrm{d}v}{\mathrm{d}y}\right)^2 + v^2\frac{\mathrm{d}^2 v}{\mathrm{d}y^2}$$

Did you see the mistake? It occured in (7) when $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}y}$ was switched around. One may say that incorrect results will not be reached if such invalid manipulations are avoided, but the notation encourages such manipulations. Plus, the derivation above can hardly be described as readable. Writing out so many fractions becomes quite tedious. For this reason, an alternative is desired.

I do not mean to imply that Leibniz notation is "wrong", merely that it is easy to make mistakes without careful thought.

2 A functional alternative

The lagrange notation for differentiation, $f'(x)$, only works as an operator on functions. So we need a way to get functions relating different variables in order to do related rates. Suppose we have a relation between two variables, y and x,

$$y = f(x)$$

then the notation y_x (the function which returns y given the value of x) is the function f. So $y_x = f$. We also have

$$y_x = y|_{x=I}$$

where I is the identity function. It is worth clearing up some common confusions. First, the following method for defining a function

$$g(x) = x^2$$

is in fact shorthand for the following:

$$g(x) = x^2 \quad \forall x$$

There is always an implicit $\forall x$. So x is not a particular value. But x is also not a true, independent (free) variable, but is instead a "dummy" or "bound" variable, a meaningless letter we use so that we can define a function in terms of an algebraic expression. We could define the function using a different letter,

$$g(a) = a^2 \quad \forall a$$

which would define the *same* function. So the function is not inherently a function of x. We could also dispense with variables and simply define the function directly as

$$g = I^2$$

where I is the identity function.

3 Identities

1. In this notation system, we will use $f\langle x\rangle$ instead of $f(x)$ to make the difference between function evaluation and multiplication clear. We will also extend evaluation to work with function composition:

   $$f\langle g\rangle = f \circ g$$

2. If $y_x = f$ then $x_y = f^{-1}$. Or more succinctly,

$$y_x^{-1} = x_y$$

Note that this is only strictly accurate if y_x is an injective function. The validity of this notation for non-injective functions will be discussed later.

3. Inverse function identity:

$$y_x\langle x_y\rangle = I$$

4. Changing the variable:

$$y_x\langle x_t\rangle = y_t$$

5. Composition with the identity function:

$$f\langle I\rangle = f$$

6. The derivative of y with respect to x is y'_x. This is equivalent to

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=I}$$

7. The derivative of the inverse:

$$y'_x = \left(x_y^{-1}\right)' = \frac{1}{x'_y\langle x_y^{-1}\rangle} = \frac{1}{x'_y\langle y_x\rangle}$$

8. The chain rule:

$$f\langle g\rangle' = g' f'\langle g\rangle$$

4 Applications

4.1 Related Rates - First Derivative

Let's start with the canonical related rates problem:

$$x = f(t)$$

$$y = g(t)$$

$$y'_x = y_t\langle t_x\rangle' = t'_x y'_t\langle t_x\rangle = \frac{y'_t\langle t_x\rangle}{x'_t\langle t_x\rangle} \tag{12}$$

Making it a function of t by composition with x_t,

$$y'_x\langle x_t\rangle = \frac{y'_t}{x'_t} = \frac{g'}{f'} \tag{13}$$

This is slightly longer, and more alien, but we will make up for it in more advanced problems.

4.2 Related Rates - Second Derivative, method 1

Let's find the second derivative of y with respect to x. Two methods will be demonstrated. First, by differentiating (12) and then making it into a function of t:

$$y''_x = \left(\frac{y'_t\langle t_x\rangle}{x'_t\langle t_x\rangle}\right)' = \left(\frac{g'\langle t_x\rangle}{f'\langle t_x\rangle}\right)' = t'_x\left(\frac{f'g''-f''g'}{(f')^2}\right)\langle t_x\rangle$$

Notice the trick used of composition of the large function in parentheses by t_x instead of repeating t_x inside each f and g. This trick can be used throughout calculus when not using variables.

$$y''_x = \frac{1}{x'_t\langle t_x\rangle}\left(\frac{f'g''-f''g'}{(f')^2}\right)\langle t_x\rangle = \left(\frac{f'g''-f''g'}{(f')^3}\right)\langle t_x\rangle$$

Making it a function of t is easily seen by inspection, but I will write it out:

$$y''_x\langle x_t\rangle = \frac{f'g''-f''g'}{(f')^3}$$

4.3 Related Rates - Second Derivative, method 2

Now we will differentiate (13) to directly get the second derivative as a function of t:

$$(y'_x\langle x_t\rangle)' = \left(\frac{g'}{f'}\right)'$$

$$x'_t y''_x\langle x_t\rangle = \frac{f'g''-f''g'}{(f')^2}$$

$$y''_x\langle x_t\rangle = \frac{f'g''-f''g'}{(f')^3}$$

4.4 Differential Equation Substitution

Now let's redo the differential equation problem:

$$y'''_t = f\langle y_t\rangle$$

$$v_t = y'_t \tag{14}$$

$$y''_t = v'_t = v_y\langle y_t\rangle' = y'_t v'_y\langle y_t\rangle = v_t v'_y\langle y_t\rangle \tag{15}$$

$$y'''_t = \left(v_t v'_y\langle y_t\rangle\right)' = v'_t v'_y\langle y_t\rangle + v_t y'_t v''_y\langle y_t\rangle$$

Substituting (14) and (15):

$$y'''_t = \left(v_t v'_y\langle y_t\rangle\right) v'_y\langle y_t\rangle + v_t^2 v''_y\langle y_t\rangle$$

$$y'''_t = v_t\left(v'_y\langle y_t\rangle\right)^2 + v_t^2 v''_y\langle y_t\rangle$$

After substituting this into the differential equation, we would change everything to be a function of y using t_y,

$$y_t'''\langle t_y\rangle = v_y\left(v_y'\right)^2 + v_y^2 v_y'' \tag{16}$$

This is the correct equation. This derivation is much more compact than the derivation using Leibniz notation. As an aside, our differential equation would end up being

$$v_y\left(v_y'\right)^2 + v_y^2 v_y'' = f$$

5 Multiple variables

This notation could also work for multivariate functions. We need only augment the identity function and lagrange notation for the derivative. The function I_n is the function which returns the nth argument unchanged. That is,

$$I_n\langle x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m\rangle = x_n \quad \forall x_1 \ldots x_m$$

We define $\partial_n f$ to be the partial derivative of the multivariate function f with respect to the nth argument. So we have

$$\partial_n I_k = 0 \quad n \neq k$$

$$\partial_n I_n = 1$$

For a function, f, of a single variable, we have

$$\partial_1 f = f'$$

If we have

$$x = f\langle u, v\rangle$$

$$y = g\langle u, v\rangle$$

we can extend the notation (I think...). For instance, y_{xu} is the function which gives y in terms of x and u. The derivative of y with respect to x is any of the following

$$\partial_1 y_{xu}$$

$$\partial_1 y_{xv}$$

$$\partial_2 y_{ux}$$

$$\partial_2 y_{vx}$$

We also have the following messy generalization for changing a single variable:

$$y_{x_1 x_2 \ldots x_n \ldots x_m}\langle I_1, I_2, \ldots, I_{n-1}, (x_n)_{x_1 x_2 \ldots v \ldots x_m}, I_{n+1}, \ldots, I_m\rangle = y_{x_1 x_2 \ldots v \ldots x_m}$$

where v is another variable involved in the relational map.

Here's an example of this:

$$x_{vy}\langle I_1, y_{vw}\rangle = x_{vw}$$

or

$$x_{vy}\langle t_1, y_{vw}\langle t_1, t_2\rangle\rangle = x_{vw}\langle t_1, t_2\rangle \quad \forall t_1, t_2$$

We also need to generalize the inverse. The operator N_i ("iNverse") will give the inverse of the function with respect to the ith variable. So

$$f\langle N_1 f\langle x, y\rangle, y\rangle = x$$

$$f\langle x, N_2 f\langle x, y\rangle\rangle = y$$

In a variable mapping function, this operator will swap the leading variable with the ith variable:

$$N_i y_{x_1 x_2 \dots x_n} = x_{i x_1 x_2 \dots x_{i-1} y x_{i+1} \dots x_n}$$

For example,

$$N_1 v_{xy} = x_{vy}$$

$$N_2 v_{xy} = y_{xv}$$

6 Non-injective Functions and Formalization

One way to deal with non-injective functions is to define y_x not as a function, but as a new type of object, an "omnipotent" function which is aware of the current value of y when making small perturbations to x. More formally, y_x is restricted to a certain branch of the function, and is not ready to be used until we take its derivative, y'_x and rewrite it in terms of the necessary variables. So it is an object which will be in a non-evaluatable state until transformations are performed on it that enable an unambiguous evaluation. This should enable formally correct calculations of derivatives for such relations as $x^2 + y^2 = r^2$. I am not skilled enough to make this any more formal, so this hand-wavey expanation will have to do. Demonstrating an isomorphism between this notation and Leibniz notation should be sufficient to prove it yields correct results at the end.

7 An Informal Derivation of the Jacobian

We wish to rewrite the integral

$$\int_a^b \int_{p\langle y\rangle}^{q\langle y\rangle} f\langle x, y\rangle \,\mathrm{d}x \,\mathrm{d}y$$

using the variables v, w in the relation

$$x = g\langle v, w\rangle$$

$$y = h\langle v, w\rangle$$

In contrast to how a change of variables is typically done with the Leibniz notation for integration, we will use the identity

$$\int_a^b f\langle x\rangle \,\mathrm{d}x = \int_{g^{-1}\langle a\rangle}^{g^{-1}\langle b\rangle} g'\langle x\rangle f\langle g\langle x\rangle\rangle \,\mathrm{d}x$$

Notice that we do not necessarily change the variable of integration. In this case, x is just being used as a dummy (binding) variable rather than a free variable as is customary in Leibniz notation. However, to keep things semantically meaningful, we will use $\tilde{x}, \tilde{y}, \tilde{v}, \tilde{w}$ as the binding variables. These are not the same as the x, y, v, w variables, but we give them a similar name to keep the semantic meaning of the integral straight while performing the transformations. We will also cast f as a variable rather than a function to make things more compact. So we have

$$\int_a^b \int_{p\langle\tilde{y}\rangle}^{q\langle\tilde{y}\rangle} f_{xy}\langle\tilde{x}, \tilde{y}\rangle \,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}$$

First we change $\tilde{x}$ to $\tilde{v}$:

$$\int_a^b \int_{v_{xy}\langle p\langle\tilde{y}\rangle,\tilde{y}\rangle}^{v_{xy}\langle q\langle\tilde{y}\rangle,\tilde{y}\rangle} \partial_1 x_{vy}\langle\tilde{v}, \tilde{y}\rangle f_{xy}\langle x_{vy}\langle\tilde{v}, \tilde{y}\rangle, \tilde{y}\rangle \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{y}$$

$$\int_a^b \int_{v_{xy}\langle p\langle\tilde{y}\rangle,\tilde{y}\rangle}^{v_{xy}\langle q\langle\tilde{y}\rangle,\tilde{y}\rangle} \partial_1 x_{vy}\langle\tilde{v}, \tilde{y}\rangle f_{vy}\langle\tilde{v}, \tilde{y}\rangle \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{y}$$

Now we change $\tilde{y}$ to $\tilde{w}$:

$$\int\limits_{w_{vy}\langle\tilde{v},\, a\rangle}^{w_{vy}\langle\tilde{v},\, b\rangle} \partial_2 y_{vw}\langle\tilde{v}, \tilde{w}\rangle \int\limits_{v_{xy}\langle p\langle y_{vw}\langle\tilde{v},\,\tilde{w}\rangle\rangle,\, y_{vw}\langle\tilde{v},\,\tilde{w}\rangle\rangle}^{v_{xy}\langle q\langle y_{vw}\langle\tilde{v},\,\tilde{w}\rangle\rangle,\, y_{vw}\langle\tilde{v},\,\tilde{w}\rangle\rangle} \partial_1 x_{vy}\langle\tilde{v}, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle f_{vy}\langle\tilde{v}, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{w}$$

One may immediately object that the integration variables are in the integration bounds in an invalid manner. Specifically, $\tilde{v}$ is in the bounds of the outermost integral even though $\tilde{v}$ is used as the integration variable in the innermost integral. And even worse, $\tilde{v}$ is in the integration bounds of the very integral which uses it as an integration variable. This will be addressed later.

$$\int_{w_{vy}\langle\tilde{v},a\rangle}^{w_{vy}\langle\tilde{v},b\rangle} \int\limits_{v_{xy}\langle\ldots\rangle}^{v_{xy}\langle\ldots\rangle} \partial_2 y_{vw}\langle\tilde{v}, \tilde{w}\rangle \partial_1 x_{vy}\langle\tilde{v}, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle f_{vw}\langle\tilde{v}, \tilde{w}\rangle \,\mathrm{d}\tilde{v}\,\mathrm{d}\tilde{w}$$

For compactness, let ... be shorthand for $\tilde{v}, \tilde{w}$

$$\iint_D \partial_2 y_{vw}\langle...\rangle \partial_1 x_{vy}\langle\tilde{v}, y_{vw}\langle...\rangle\rangle f_{vw}\langle...\rangle \,\mathrm{d}\tilde{v}\mathrm{d}\tilde{w}$$

We want $\partial_1 x_{vy}\langle\tilde{v}, y_{vw}\langle...\rangle\rangle$ to be rewritten with functions in terms of v, w directly, rather than as a composition which makes it in terms of v, w. We will need to use the equivalent identitites for $\left(f^{-1}\right)'(x)$ but generalized for partial derivatives. We will derive the necessary identities on the spot:

$$x_{vy}\langle\tilde{v}, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle = x_{vw}\langle\tilde{v}, \tilde{w}\rangle$$

Differentiating with respect to $\tilde{v}$,

$$\partial_1 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle + \partial_1 y_{vw}\langle ...\rangle \partial_2 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle = \partial_1 x_{vw}\langle ...\rangle$$

$$\partial_1 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle = \partial_1 x_{vw}\langle ...\rangle - \partial_1 y_{vw}\langle ...\rangle \partial_2 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle$$

$$\iint_D \partial_2 y_{vw}\langle ...\rangle \left[\partial_1 x_{vw}\langle ...\rangle - \partial_1 y_{vw}\langle ...\rangle \partial_2 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle\right] f_{vw}\langle ...\rangle \, \mathrm{d}\tilde{v}\mathrm{d}\tilde{w}$$

We still have a pesky $\partial_2 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle$, but this one is not a "mixed" partial (partial of first argument, with inverse of second argument inside), since it is the partial of the second argument, with the inverse of the second argument inside. For this reason, we expect that we can truly get rid of it (and that we won't just get a similar mixed partial identity that brings us back to square one).

$$x_{vy}\langle\tilde{v}, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle = x_{vw}\langle\tilde{v}, \tilde{w}\rangle$$

Differentiating with respect to $\tilde{w}$,

$$\partial_2 y_{vw}\langle ...\rangle \partial_2 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle = \partial_2 x_{vw}\langle ...\rangle$$

$$\partial_2 x_{vy}\langle\tilde{v}, y_{vw}\langle ...\rangle\rangle = \frac{\partial_2 x_{vw}\langle ...\rangle}{\partial_2 y_{vw}\langle ...\rangle}$$

Substituting,

$$\iint_D \partial_2 y_{vw}\langle ...\rangle \left[\partial_1 x_{vw}\langle ...\rangle - \partial_1 y_{vw}\langle ...\rangle \frac{\partial_2 x_{vw}\langle ...\rangle}{\partial_2 y_{vw}\langle ...\rangle}\right] f_{vw}\langle ...\rangle \, \mathrm{d}\tilde{v}\mathrm{d}\tilde{w}$$

$$\iint_D \left[\partial_2 y_{vw} \partial_1 x_{vw} - \partial_1 y_{vw} \partial_2 x_{vw}\right]\langle ...\rangle f_{vw}\langle ...\rangle \, \mathrm{d}\tilde{v}\mathrm{d}\tilde{w}$$

$$\iint_D f_{vw}\langle ...\rangle \begin{vmatrix} \partial_1 x_{vw} & \partial_1 y_{vw} \\ \partial_2 x_{vw} & \partial_2 y_{vw} \end{vmatrix} \langle ...\rangle \, \mathrm{d}\tilde{v}\mathrm{d}\tilde{w}$$

With the bounds,

$$\int\limits_{w_{vy}\langle\tilde{v}, a\rangle}^{w_{vy}\langle\tilde{v}, b\rangle} \int\limits_{v_{xy}\langle p\langle y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle}^{v_{xy}\langle q\langle y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle, y_{vw}\langle\tilde{v}, \tilde{w}\rangle\rangle} f_{vw}\langle\tilde{v}, \tilde{w}\rangle \begin{vmatrix} \partial_1 x_{vw} & \partial_1 y_{vw} \\ \partial_2 x_{vw} & \partial_2 y_{vw} \end{vmatrix} \langle\tilde{v}, \tilde{w}\rangle \, \mathrm{d}\tilde{v} \, \mathrm{d}\tilde{w}$$

7.1 The bounds issue

Note: Upon further experimentation, I have found that the methodology outlined in this section is still not sufficient to formalize the bounds. It was not able to produce consistent/clear results for integrating over a circular disk. I'm optimistic more investigation into this problem could alleviate this; for now, consider this section a prototype/proof of concept.

The question is how can these bounds be interpreted? One way to interpret them is to observe that

$$\int_a^b f\langle x\rangle \, \mathrm{d}x$$

is equivalent to

$$\int_{x=a}^{x=b} f\langle x\rangle \, \mathrm{d}x$$

So that if we have

$$\int_{a\langle x\rangle}^{b\langle x\rangle} f\langle x\rangle \, \mathrm{d}x$$

then the bounds must satisfy

$$\int_{x=a\langle x\rangle}^{x=b\langle x\rangle} f\langle x\rangle \, \mathrm{d}x$$

For example,

$$\int_0^{2x+5} f\langle x\rangle \, \mathrm{d}x = \int_{x=0}^{x=2x+5} f\langle x\rangle \, \mathrm{d}x$$

$$x = 2x + 5 \implies x = -5$$

$$\int_0^{2x+5} f\langle x\rangle \, \mathrm{d}x = \int_0^{-5} f\langle x\rangle \, \mathrm{d}x$$

If there are multiple solutions, then all combinations are put into different integrals which get added up,

$$\int_0^{x^3} f\langle x\rangle \, \mathrm{d}x$$

$$x = x^3 \implies x = 0 \text{ or } x = 1 \text{ or } x = -1$$

$$\int_0^{x^3} f\langle x\rangle \, \mathrm{d}x = \int_0^{0} f\langle x\rangle \, \mathrm{d}x + \int_0^{1} f\langle x\rangle \, \mathrm{d}x + \int_0^{-1} f\langle x\rangle \, \mathrm{d}x$$

Note that,

$$\int_0^{x} f\langle x\rangle \, \mathrm{d}x$$

is undefined, because $x = x$ has infinitely many solutions.

An informal justification for the interpretation given can be sketched out. Imagine that a computer were to evaluate the integral

$$\int_{a\langle x\rangle}^{b\langle x\rangle} f\langle x\rangle \,\mathrm{d}x$$

The computer would sweep through values in the range $a\langle x\rangle, b\langle x\rangle$, evaluating the function at those values and adding them up. But through which range to sweep is not clear since $a\langle x\rangle$ and $b\langle x\rangle$ aren't numbers. Instead, the sweeping can be done with a separate parameter, $0 < t < 1$. The value of x in terms of t would then be $a\langle x\rangle + t(b\langle x\rangle - a\langle x\rangle)$. This isn't a number either, but for particular values of t, such as 0.5, the computer would then evaluate the function as such:

$$f\langle x\rangle|_{x=a\langle x\rangle+0.5(b\langle x\rangle-a\langle x\rangle)}$$

and the value of x would be solved at evaluation time. Since this might have multiple solutions, the integration could be done on each branch of the function (of t),

$$p\langle t\rangle = [a\langle I\rangle + t(b\langle I\rangle - a\langle I\rangle) - I]^{-1}\langle 0\rangle$$

separately. That is, the computer might choose to be consistent with which solution it chooses. If the equation became $x^2 = t$, then it would always choose $x = \sqrt{t}$ for one of the integrals, and $x = -\sqrt{t}$ for the other.

In fact, the computer would not *have* to be consistent, since it is simply adding things up. It could choose the branch randomly. But it would have to make the opposite choices in the second integration. Let's test this interpretation on an example to see if it gives consistent results. Switching the order of integration (for nice functions) should make no difference under this interpretation.

$$\int_5^x \int_0^y y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^{x-5} \int_0^{y+5} (y+5)^2 \,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \int_{x=0}^{x=(x-5)y+5} (x-5)\left((x-5)y+5\right)^2 \,\mathrm{d}x\,\mathrm{d}y$$

$$x = (x-5)y+5 \implies x = \frac{-5y+5}{1-y}$$

$$\int_5^x \int_0^y y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{\frac{-5y+5}{1-y}} (x-5)\left((x-5)y+5\right)^2 \,\mathrm{d}x\,\mathrm{d}y = -156.25$$

Now let's switch the order and see if we get the same answer,

$$\int_0^y \int_5^x y^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_5^{yx} yy^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{y=5}^{y=yx} y^3 \,\mathrm{d}y\,\mathrm{d}x$$

$$y = yx, x \neq 1 \implies y = 0$$

$x \neq 1$ as long as we exclude that single point from the integration, which won't change the result.

$$\int_0^y \int_5^x y^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_5^0 y^3 \,\mathrm{d}y\,\mathrm{d}x = -156.25$$

8 Other remarks

We can perform calculus entirely without variables[2]:

$$f = I^3$$

$$f' = 3I^2$$

$$f = \int_0^I 3I^2$$

$$f = \int_0^I \int_0^I 6I$$

The indefinite integral:

$$F = C + \int^I f$$

where C is a constant function.

Here is an example of calculating a derivative for $x^2 + y^2 = r^2$. Let us calculate the derivative of x with respect to y:

$$x_y^2 + I^2 = r^2$$

$$2x_y x_y' + 2I = 0$$

$$x_y' = -\frac{I}{x_y}$$

So if $y = 3$ and $x = 4$ (implying $r = 5$), then

$$x_y'\langle 3\rangle = -\frac{3}{4}$$

Since this relation is non-injective, this isn't simply a function being evaluated at a value of 3. There's more magic going on in the background.

[2]For a larger exposition on variableless calculus, see "Alternative mathematical notation and its applications in calculus" by Jakub Marian: `https://jakubmarian.com/data/bachelor_thesis.pdf`