Constructing the Cubic Formula

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1 Quadratic Background

The standard method to derive the quadratic formula is completing the square:

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

At this stage, a square root can be taken to get the full formula. Note that this shows how every quadratic can be written as a series of transformations on the function x^2. We can also multiply by a to ensure we are writing the original quadratic:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c$$

Recall that $-\frac{b}{2a}$ is the center of the quadratic, the x value where the minimum is. The addition of $\frac{b}{2a}$ therefore represents the off-centering of the quadratic. We can recenter it by making the substitution $(x - \frac{b}{2a}) \to x$:

$$a\left(\left(x - \frac{b}{2a}\right) + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c = ax^2 - a\left(\frac{b}{2a}\right)^2 + c$$

or, showing it happening in the original

$$a\left(x - \frac{b}{2a}\right)^2 + b\left(x - \frac{b}{2a}\right) + c = ax^2 - \frac{b^2}{4a} + c$$

This is symmetric because there is no x term, which means that we have $P(-x) = P(x)$, meaning the quadratic is centered about the y-axis.

2 Depressed cubic

However, it is not possible to write every cubic as transformations of x^3, so we cannot simply complete the cube. However, we can center it to simplify it and remove the x^2 term. The center of a cubic is $-\frac{b}{3a}$ (if one did not know this, one could write it as a variable δ, and then solve for δ by setting the coefficient on x^2 to 0):

$$x = x' - \frac{b}{3a} \tag{1}$$

$$a\left(x' - \frac{b}{3a}\right)^3 + b\left(x' - \frac{b}{3a}\right)^2 + c\left(x' - \frac{b}{3a}\right) + d = 0$$
$$ax'^3 + \left(c - \frac{b^2}{3a}\right)x' + \frac{2b^3 - 9abc}{27a^2} + d = 0$$

Dividing both sides by a:

$$x'^3 + \left(\frac{3ac - b^2}{3a^2}\right)x' + \frac{2b^3 - 9abc}{27a^3} + \frac{d}{a} = 0 \tag{2}$$

This is a depressed cubic. Since every cubic can be written this way, we will now forget general cubics and instead attempt to solve the simpler

$$x^3 + cx + d = 0$$

While not symmetric about the y-axis, this is centered because for

$$P(x) = x^3 + cx$$

we have

$$P(-x) = -P(x)$$

The d merely moves this centered cubic up or down.

We can do even better than this. It turns out that all cubics can be written as transformations of these three primitive cubics:

$$x^3 + x$$
$$x^3$$

$$x^3 - x$$

The x^3 function will appear flat at the center. The x^3+x function will be sloped upwards at the center. The $x^3 - x$ function will be sloped downwards and have a maximum on the left and a minimum on the right. We will concentrate on this last one since it is the only one where all three roots can be distinct real numbers.

Let's use transformations to get rid of the variable c:

$$x^3 + cx + d = 0$$

We've already used up shift transformations to remove b, but we can still scale the x and y axes:

$$x = \lambda x' \tag{3}$$

And multiplying both sides by τ,

$$\tau\lambda^3 x'^3 + \tau\lambda c x' + \tau d = 0$$

$$\tau\lambda^3 = 1$$
$$\tau\lambda c = -1$$

Dividing the first by the second and rearranging,

$$\lambda^2 = -c$$
$$\lambda = \pm\sqrt{-c}$$

The positive variant will be chosen:

$$\lambda = \sqrt{-c}$$
$$\tau = \frac{-1}{c\lambda} = -\frac{1}{\sqrt{-c^3}}$$

So

$$x'^3 - x' - \frac{d}{\sqrt{-c^3}} = 0 \tag{4}$$

and our transformation was

$$x = \left(\sqrt{-c}\right) x' \tag{5}$$

This doesn't work for $c = 0$, and requires imaginary numbers for $c > 0$. This reflects that these other two cases are transformations of the other two primitive cubics.

So now we will work with the very depressed cubic

$$x^3 - x + d = 0$$

3 Root functions

Since we don't know the cubic formula yet, we will represent its various forms with functions so we can deduce what properties it must have.

The general cubic formula, $f_n(a,b,c,d)$, where $n = 1,2,3$ picks between one of three roots, of undecided ordering, satisfies:

$$af_n(a,b,c,d)^3 + bf_n(a,b,c,d)^2 + cf_n(a,b,c,d) + d = 0$$

The depressed cubic formula, $g_n(c,d) = f_n(1,0,c,d)$,

$$g_n(c,d)^3 + cg_n(c,d) + d = 0$$

The very depressed cubic formula, $f_{3n}(d) = f_n(1,0,-1,d)$:

$$f_{3n}(d)^3 - f_{3n}(d) + d = 0$$

We write these as f_{3n} to distinguish them from the f_{1n} function for $x^3 + x$ and the f_{2n} for x^3.

The following branch cuts will be chosen. When all roots are real and distinct, we will order the roots as,

$$f_{31}(d) < f_{32}(d) < f_{33}(d)$$

3.1 Transformation identities

Suppose we are modeling a physical system with quadratics or cubics. If we change the units we use to measure things, either changing the units in only the x-axis, only the y-axis, or in both, that is equivalent to scaling the axes. If we scale the x-axis by λ, then the roots should also scale by λ:

$$ax^3 + bx^2 + cx + d = 0 \implies x = f_n(a,b,c,d)$$

$$x = \lambda x'$$

$$a(\lambda x')^3 + b(\lambda x')^2 + c(\lambda x') + d = a\lambda^3 x'^3 + b\lambda^2 x'^2 + c\lambda x' + d = 0$$

$$\implies x' = f_n(\lambda^3 a, \lambda^2 b, \lambda c, d)$$

The x-scaling transform is therefore:

$$\frac{1}{\lambda} f_n(a, b, c, d) = f_n(\lambda^3 a, \lambda^2 b, \lambda c, d)$$

Since scaling the y-axis doesn't change the roots, we have y-scale invariance:

$$f_n(\lambda a, \lambda b, \lambda c, \lambda d) = f_n(a, b, c, d)$$

Combining these gives us a variation on the x-scaling identity:

$$\lambda f_n(a, b, c, d) = f_n(a, \lambda b, \lambda^2 c, \lambda^3 d)$$

Setting $a = 1, b = 0$ shows that this one also applies to the depressed root function:

$$\lambda g_n(c, d) = g_n(\lambda^2 c, \lambda^3 d) \tag{6}$$

We can use similar logic to get a shift transform, but we won't be using it:

$$\delta + f_n(a, b, c, d) = f_n(a, b - 3a\delta, c - 2b\delta + 3a\delta^2, d - c\delta + b\delta^2 - a\delta^3)$$

3.2 The general cubic formula in terms of the depressed formulas

Now we will write the general cubic formula $f_n(a, b, c, d)$ in terms of the depressed varieties.

$$ax^3 + bx^2 + cx + d = 0$$
$$x = f_n(a, b, c, d)$$

According to the shift transformation (equation 1) we used to get the depressed cubic,

$$x = x' - \frac{b}{3a}$$

and (by equation 2),

$$x' = g_m\left(\frac{3ac - b^2}{3a^2}, \frac{2b^3 - 9abc}{27a^3} + \frac{d}{a}\right)$$

Therefore,

$$f_n(a, b, c, d) = -\frac{b}{3a} + g_m\left(\frac{3ac - b^2}{3a^2}, \frac{2b^3 - 9abc}{27a^3} + \frac{d}{a}\right)$$

We are not worrying about the ordering/correspondence of the roots. Using the x-scale transform (equation 6) to pull out a factor of $\frac{1}{3a}$,

$$f_n(a,b,c,d) = \frac{-b + g_m\left(9ac - 3b^2, 2b^3 - 9abc + 27a^2d\right)}{3a} \tag{7}$$

Now we will write g in terms of the very depressed root function f_{3n}.

$$x^3 + cx + d = 0$$
$$x = g_n(c,d)$$

Recall equations 4 and 5,

$$x'^3 - x' - \frac{d}{\sqrt{-c^3}} = 0$$
$$x = \left(\sqrt{-c}\right) x'$$

$$x' = f_{3m}\left(-\frac{d}{\sqrt{-c^3}}\right)$$
$$g_n(c,d) = \sqrt{-c} f_{3m}\left(-\frac{d}{\sqrt{-c^3}}\right) \tag{8}$$

Combining equations 7 and 8,

$$f_n(a,b,c,d) = \frac{-b + \sqrt{3b^2 - 9ac} f_{3m}\left(\frac{2b^3 - 9abc + 27a^2d}{(\sqrt{3b^2 - 9ac})^3}\right)}{3a} \tag{9}$$

This is valid for $b^2 - 3ac > 0$

4 Related root identities

In investigating the problem of solving the cubic equation, one of the first things to do is examine the special cases. For instance, one special case is when the cubic can be written as transformations of x^3. In this case, it can be solved by completing the cube. Another special case is when

$$\frac{b}{a} = \frac{d}{c} = \lambda$$

in which case the cubic can be written as

$$ax^2(x + \lambda) + c(x + \lambda) = 0$$

And solved with the quadratic formula. $\frac{b}{c} = \frac{a}{d}$ is similar. The case of $d = 0$ is another special case. However, all of these simple cases can be derived from equation 7 and/or tell us nothing useful about f_{3n}.

One special case is when two of the roots are equal:

$$(x - r_1)(x - r_2)(x - r_2) = x^3 - x + d = 0$$

For the very depressed cubic, this means that one of the maxima is touching the x-axis. We can solve for $r_1 = \pm\frac{1}{\sqrt{3}}$ and $r_2 = \mp\frac{2}{\sqrt{3}}$. Since we can also write d in terms of r_1 and r_2. For the negative r_1 case, we learn

$$f_{31}\left(\frac{2}{\sqrt{3^3}}\right) = -\frac{1}{\sqrt{3}}$$
$$f_{32}\left(\frac{2}{\sqrt{3^3}}\right) = f_{33}\left(\frac{2}{\sqrt{3^3}}\right) = \frac{2}{\sqrt{3}}$$

This is better, but still no help. We were just able to solve the special case where $r_3 = r_2$. What about the special case where $r_3 = 2r_2$? Or more generally, where $r_3 = kr_2$?

$$(x - r_1)(x - r_2)(x - kr_2) = x^3 - x + d = 0$$

This doesn't seem like it would help, since we are surely just replacing the unknown quantity r_3 with the equally unknown k, but it does[1]. First, expand,

$$x^3 - (r_1 + (k+1)r_2)x^2 + \left((k+1)r_1r_2 + kr_2^2\right)x - kr_1r_2^2 = x^3 - x + d = 0$$

Since these are the roots of a very depressed cubic, they must be such that the coefficient on the quadratic term is 0 and on the linear term is -1:

$$r_1 + (k+1)r_2 = 0 \implies r_1 = -(k+1)r_2$$
$$(k+1)r_1r_2 + kr_2^2 = -1$$
$$-(k+1)^2r_2^2 + kr_2^2 = -1$$
$$\left[k - (k+1)^2\right]r_2^2 = -1$$

[1]In most cases where you try something like this on a cubic, in the process of solving for the necessary free parameters, you will be given back the cubic you are trying to solve, or one equally or more difficult. For instance, if you had the clever idea to try writing cubics as the difference of two squared quadratics,

$$(ax^2 + b_1x + c_2)^2 - (ax^2 + b_2x + c_2)^2 = x^3 - x + d = 0$$

or some similar idea, then in order to determine the free parameters a, b_1, b_2, c_1, c_2, you will be required to solve not a nice cube equation $p^3 + d = 0$, but instead either something that looks like $p^3 + p + d = 0$, or even worse, a sixth degree polynomial.

However, this doesn't appear to happen with this trick, at least not fully.

$$r_2 = \sqrt{\frac{1}{(k+1)^2 - k}} = \frac{1}{\sqrt{k^2+k+1}}$$

$$r_1 = -(k+1)r_2 = \frac{-(k+1)}{\sqrt{k^2+k+1}}$$

$$r_3 = kr_2 = \frac{k}{\sqrt{k^2+k+1}}$$

$$d = -kr_1r_2^2 = -k\left(\frac{-(k+1)}{\sqrt{k^2+k+1}}\right)\left(\frac{1}{\sqrt{k^2+k+1}}\right)^2 = \frac{k^2+k}{(\sqrt{k^2+k+1})^3}$$

If $k > 1$, then $r_1 < r_2 < r_3$, so

$$f_{31}\left(\frac{k^2+k}{(\sqrt{k^2+k+1})^3}\right) = \frac{-(k+1)}{\sqrt{k^2+k+1}}$$

$$f_{32}\left(\frac{k^2+k}{(\sqrt{k^2+k+1})^3}\right) = \frac{1}{\sqrt{k^2+k+1}}$$

$$f_{33}\left(\frac{k^2+k}{(\sqrt{k^2+k+1})^3}\right) = \frac{k}{\sqrt{k^2+k+1}}$$

Additionally, $k > 1$ implies $0 < d < \frac{2}{\sqrt{3^3}}$.

Now, we cannot simply solve for k in terms of d to get the formula. If we tried that, we would have to solve a sextic (sixth degree polynomial). Instead, we are going to try to construct, or at the very least guess, the function $f_{32}(d)$ necessary to transform the expression

$$d = \frac{k^2+k}{(\sqrt{k^2+k+1})^3}$$

into

$$f_{32}(d) = \frac{1}{\sqrt{k^2+k+1}}$$

The first idea is to take a cube root on d so the denominators match. But this gives us a cube root at the top which encloses the numerator, preventing further changes to it. Our only other option is to square d, make necessary changes to the numerator, and then take a square root and cube root later to fix the denominator.

$$d^2 = \frac{(k^2+k)^2}{(k^2+k+1)^3} = \frac{k^4+2k^3+k^2}{k^6+3k^5+6k^4+7k^3+6k^2+3k+1}$$

We can make changes to the numerator by adding constants, which will add multiples of the denominator to the numerator. Adding d itself probably won't help, since d has a square root in its denominator, which would mean a square root gets added to the numerator. However, instead of adding multiples of the large sextic to the quartic, we will want to do it the other way around for simplicity, so that multiples of the quartic get added to the sextic in the numerator:

$$1 + \lambda d^2 = \frac{(k^6 + 3k^5 + 6k^4 + 7k^3 + 6k^2 + 3k + 1) + \lambda(k^4 + 2k^3 + k^2)}{k^6 + 3k^5 + 6k^4 + 7k^3 + 6k^2 + 3k + 1}$$
$$1 + \lambda d^2 = \frac{k^6 + 3k^5 + (6+\lambda)k^4 + (7+2\lambda)k^3 + (6+\lambda)k^2 + 3k + 1}{(k^2 + k + 1)^3}$$

Now we want to use our free parameter λ to make the sextic on top into either a perfect cube of a quadratic or a perfect square of a cubic, so that we can take the cube root or square root, respectively. We hope that there will be two or more solutions to λ, so that after taking the root, we can create further cancellations on the numerator.

It turns out there is no possible lambda that can make it into a perfect cube. So we will hope it is possible to make it into a perfect square:

$$\begin{aligned} k^6 + 3k^5 + (6+\lambda)k^4 + (7+2\lambda)k^3 + (6+\lambda)k^2 + 3k + 1 \\ = (k^3 + bk^2 + ck + e)^2 \\ = k^6 + 2bk^5 + (b^2+2c)k^4 + (2e+2bc)k^3 + (c^2+2be)k^2 + 2cek + e^2 \end{aligned}$$

#	Equation
1	$2b = 3$
2	$b^2 + 2c = 6 + \lambda$
3	$2e + 2bc = 7 + 2\lambda$
4	$c^2 + 2be = 6 + \lambda$
5	$2ce = 3$
6	$e^2 = 1$

This system is overdetermined, so we need to get lucky. Equations 1, 5, and 6 will be used to solve for b, c, and e. Equation 2 will be used to solve for λ. Then equations 3 and 4 must be checked.

$$b = \frac{3}{2}$$

$$2ce = 3 \implies 2ce^2 = 3e \implies 2c = 3e \implies c = \frac{3}{2}e$$

$$e^2 = 1$$

$e = 1$	$e = -1$
$c = \frac{3}{2}$ Equation 2: $\lambda = b^2 + 2c - 6$ $\lambda = \frac{9}{4} - 3$ $\lambda = -\frac{3}{4}$ Check equation 3: $2e + 2bc = 7 + 2\lambda$ $2 + 2\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) \stackrel{?}{=} 7 - 2\left(\frac{3}{4}\right)$ $\frac{9}{2} + \frac{3}{2} \stackrel{?}{=} 5$ $6 = 5$ ✗	$c = -\frac{3}{2}$ Equation 2: $\lambda = b^2 + 2c - 6$ $\lambda = \frac{9}{4} - 9$ $\lambda = -\frac{27}{4}$ Check equation 3: $2e + 2bc = 7 + 2\lambda$ $-2 + 2\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right) \stackrel{?}{=} 7 - 2\left(\frac{27}{4}\right)$ $-\frac{9}{2} + \frac{27}{2} \stackrel{?}{=} 9$ $9 = 9$ ✓ Check equation 4: $c^2 + 2be = 6 + \lambda$ $\frac{9}{4} - 3 = 6 - \frac{27}{4}$ $\frac{36}{4} = 9$ ✓

The required parameters are

$$b = \frac{3}{2}, c = -\frac{3}{2}, e = -1, \lambda = -\frac{27}{4}$$

We therefore have

$$\sqrt{1 - \frac{27}{4}d^2} = \sqrt{\frac{(k^3 + \frac{3}{2}k^2 - \frac{3}{2}k - 1)^2}{(k^2 + k + 1)^3}} = \frac{|k^3 + \frac{3}{2}k^2 - \frac{3}{2}k - 1|}{(\sqrt{k^2 + k + 1})^3}$$

For $k > 1$, the numerator is always positive, so we can drop the absolute value. Recall that

$$d = \frac{k^2 + k}{(\sqrt{k^2 + k + 1})^3}$$

The only option now is to add multiples of d to the equation to make the numerator into a perfect cube.

$$\sqrt{1-\frac{27}{4}d^2}+\tau d=\frac{(k^3+\frac{3}{2}k^2-\frac{3}{2}k-1)+\tau(k^2+k)}{(\sqrt{k^2+k+1})^3}$$

$$\sqrt{1-\frac{27}{4}d^2}+\tau d=\frac{k^3+(\frac{3}{2}+\tau)k^2+(-\frac{3}{2}+\tau)k-1}{(\sqrt{k^2+k+1})^3}=\frac{(k+\beta)^3}{(\sqrt{k^2+k+1})^3}$$

$$=\frac{k^3+3\beta k^2+3\beta^2k+\beta^3}{(\sqrt{k^2+k+1})^3}$$

$$3\beta=\frac{3}{2}+\tau$$

$$3\beta^2=-\frac{3}{2}+\tau$$

$$\beta^3=-1$$

Multiplying the first two equations together:

$$9\beta^3=-\frac{9}{4}+\tau^2$$

$$-9=-\frac{9}{4}+\tau^2$$

$$\tau=\pm\sqrt{\frac{9-36}{4}}=\pm\sqrt{\frac{-27}{4}}=\pm\frac{3i\sqrt{3}}{2}$$

$$\beta=\frac{1}{2}+\frac{\tau}{3}=\frac{1\pm i\sqrt{3}}{2}$$

This is negative one times a root of unity, so the system of equations is consistent.

$$\sqrt{1-\frac{27}{4}d^2}\pm\frac{3i\sqrt{3}}{2}d=\frac{\left(k+\frac{1\pm i\sqrt{3}}{2}\right)^3}{(\sqrt{k^2+k+1})^3}$$

$$p=\sqrt[3]{\sqrt{1-\frac{27}{4}d^2}+\frac{3i\sqrt{3}}{2}d}=\frac{k+\frac{1+i\sqrt{3}}{2}}{\sqrt{k^2+k+1}}$$

$$-n=\sqrt[3]{\sqrt{1-\frac{27}{4}d^2}-\frac{3i\sqrt{3}}{2}d}=\frac{k+\frac{1-i\sqrt{3}}{2}}{\sqrt{k^2+k+1}}$$

(variables p and n added for later).

Technically we should put undetermined roots of unity in front of the cube roots, but let's ignore that and hope it works. Subtracting the second from the first gives

$$\sqrt[3]{\sqrt{1-\frac{27}{4}d^2}+\frac{3i\sqrt{3}}{2}d}-\sqrt[3]{\sqrt{1-\frac{27}{4}d^2}-\frac{3i\sqrt{3}}{2}d}=\frac{i\sqrt{3}}{\sqrt{k^2+k+1}}$$

$$\frac{\sqrt[3]{\sqrt{1-\frac{27}{4}d^2}+\frac{3i\sqrt{3}}{2}d}+\sqrt[3]{-\sqrt{1-\frac{27}{4}d^2}+\frac{3i\sqrt{3}}{2}d}}{i\sqrt{3}}=\frac{1}{\sqrt{k^2+k+1}}=\frac{p+n}{i\sqrt{3}}$$

Note $\frac{1}{i^3}=i$.

$$f_{32?}(d)=\sqrt[3]{-\frac{d}{2}+\sqrt{\frac{d^2}{4}-\frac{1}{27}}}+\sqrt[3]{-\frac{d}{2}-\sqrt{\frac{d^2}{4}-\frac{1}{27}}} \tag{10}$$

Here it is. The only uncertain thing here is whether our branch choices made sense and whether this indeed corresponds to the second branch.

Now let's add p and n in different ways to get the other branches. We know

$$p-n=\frac{2k+1}{\sqrt{k^2+k+1}}$$
$$\frac{p+n}{i\sqrt{3}}=\frac{1}{\sqrt{k^2+k+1}}$$

So

$$\frac{p-n-\frac{p+n}{i\sqrt{3}}}{2}=\frac{k}{\sqrt{k^2+k+1}}$$
$$f_{33?}(d)=\frac{(1-\frac{1}{i\sqrt{3}})p+(-1-\frac{1}{i\sqrt{3}})n}{2}$$
$$f_{33?}(d)=\frac{(-1+i\sqrt{3})p+(-1-i\sqrt{3})n}{2i\sqrt{3}}$$
$$f_{33?}(d)=u_2\sqrt[3]{-\frac{d}{2}+\sqrt{\frac{d^2}{4}-\frac{1}{27}}}+u_3\sqrt[3]{-\frac{d}{2}-\sqrt{\frac{d^2}{4}-\frac{1}{27}}}$$

where $u_2=\frac{-1+i\sqrt{3}}{2}, u_3=\frac{-1-i\sqrt{3}}{2}$ (roots of unity)

Similar steps show that

$$f_{31?}(d) = u_3 \sqrt[3]{-\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{1}{27}}} + u_2 \sqrt[3]{-\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{1}{27}}}$$

The branches do not correspond properly, however. This is likely related to the fact that we dropped the roots of unity during the derivation and used the property $-\sqrt[3]{x} = \sqrt[3]{-x}$ which is invalid for the principal cube root definition $\sqrt[3]{|z| \exp(i \arg(z))} = \sqrt[3]{|z|} \exp\left(i \frac{\arg(z)}{3}\right)$

The depressed root function is

$$g_n(c, d) = \sqrt{-c} f_{3m} \left(-\frac{d}{\sqrt{-c^3}}\right)$$

$$g_n(c, d) = \sqrt[3]{-\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{-\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$$

From here, the general cubic formula can be written, but it is ugly.

We can figure out how equation 10 works by considering the polynomial

$$x^3 - 3x + 2d = 0$$

Then

$$x = \sqrt[3]{-d + \sqrt{d^2 - 1}} + \sqrt[3]{-d - \sqrt{d^2 - 1}}$$

These terms give a value of 1 when multiplied together. Let the first term be p and the second one n. Additionally, the sum of the cubes of the terms gives $-2d$. So

$$x^3 = (p + n)^3 = p^3 + n^3 + 3p^2 n + 3pn^2 = -2d + 3p + 3n = -2d + 3x$$

$$x^3 - 3x + 2d = 0 \quad \checkmark$$

There is an extraordinary similarity with the square root denesting identity, which operates by nearly the same principles:

$$\sqrt{x + \sqrt{y}} = \sqrt{\frac{x + \sqrt{x^2 - y}}{2}} + \sqrt{\frac{x - \sqrt{x^2 - y}}{2}}$$

One potential route to the cubic formula is to therefore replace the outer square roots in the right hand side with cube roots, and then take the cube.

4.1 Obsolete lines of inquiry

4.1.1 Other root relation strategies

An attempt was also made to use the root relation $r_3 = r_2 + k$, but this was not pursued much further because the identity looked uglier than the ratio version:

$$f_{31}\left(\frac{\pm(1-k^2)\sqrt{12-3k^2}}{9}\right) = \frac{\mp 2\sqrt{12-3k^2}}{6}$$

$$f_{32}\left(\frac{\pm(1-k^2)\sqrt{12-3k^2}}{9}\right) = \frac{-3k \pm \sqrt{12-3k^2}}{6}$$

$$f_{33}\left(\frac{\pm(1-k^2)\sqrt{12-3k^2}}{9}\right) = \frac{3k \pm \sqrt{12-3k^2}}{6}$$

$$0 < k < \sqrt{3}$$

Additionally, during the root relation derivation, there was one other unpursued path:

$$d = \frac{k^2+k}{(\sqrt{k^2+k+1})^3}$$

One can square d and add constants. But if this didn't end up working, the plan was to also add multiples of d, which results in a square root added in the numerator. A square root is then placed around the entire expression. One could then check if the square roots could be denested.

4.1.2 Cleaner root functions

The quadratic equation can be made cleaner if one uses $ax^2+2bx+c=0$:

$$x = \frac{-b \pm \sqrt{b^2-c}}{a}$$

Going even further,

$$x^2+2bx+c^2 = 0 \implies x = -b \pm \sqrt{b^2-c^2}$$

We can analogously make the very depressed cubic cleaner by using $x^3-3x+d = 0$ instead. The corresponding root function is denoted $h_{3n}(d)$. It was thought this might make guessing the form easier.

4.1.3 Reciprocal transform

The reciprocal transform:

$$\frac{1}{f_n(a,b,c,d)} = f_m(d,c,b,a)$$

The root ordering is left undetermined. Unlike other transforms, this gives a non-trivial identity for the very depressed root function.

$$\frac{1}{f_{33}(d)} = \frac{1+\sqrt{3}f_{3n}\left(\frac{-2\sqrt{3}}{9}+3\sqrt{3}d^2\right)}{3d}$$

or, the cleaner varieties, with correctly determined root ordering

$0 < d < 2$	$-2 < d < 0$
$\frac{1}{h_{31}(d)} = \frac{1+h_{31}(d^2-2)}{d}$	$\frac{1}{h_{31}(d)} = \frac{1+h_{32}(d^2-2)}{d}$
$\frac{1}{h_{32}(d)} = \frac{1+h_{33}(d^2-2)}{d}$	$\frac{1}{h_{32}(d)} = \frac{1+h_{33}(d^2-2)}{d}$
$\frac{1}{h_{33}(d)} = \frac{1+h_{32}(d^2-2)}{d}$	$\frac{1}{h_{33}(d)} = \frac{1+h_{31}(d^2-2)}{d}$

It was hoped these would allow guessing the form of the formula, but that didn't work.