Lagrange Remainder for Taylor Series

James Taylor

1 Intro

The nth Taylor polynomial for a function f around c is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)(x-c)^k}{k!}$$

The nth remainder term is the difference between the Taylor polynomial and the function:

$$R_n(x) = f(x) - P_n(x)$$

The Lagrange form of the remainder gives a formula for the remainder:

$$R_n(x_0) = \frac{f^{(n+1)}(\lambda)(x_0 - c)^{n+1}}{(n+1)!}$$

for some λ between x_0 and c.

2 Intuition

Lemma 1. *If a function is continous in an interval and never takes on a certain value, then either it's always less than that value in the interval, or it's always greater than that value. Rigorously, if a function f is continuous in $[a, b]$ and $f(x) \neq h \;\; \forall x$ in $[a, b]$ then either $f(x) > h \;\; \forall x$ in $[a, b]$ or $f(x) < h \;\; \forall x$ in $[a, b]$. I have a marvelous proof of this fact, but it is too large to fit in this box.*

Suppose we're using a linear approximation of a function. We draw a line tangent to the function at $x = c$. We want to find the remainder (error) at $x = x_0$. We can write this remainder term as a quadratic correction. So we need to add a suitable acceleration to our linear approximation so that it passes through $f(x_0)$. We will call this acceleration a. So we have that

$$R_1(x) = \frac{a(x-c)^2}{2}$$

$$f(x_0) = f(c) + f'(c)(x_0 - c) + \frac{a(x_0 - c)^2}{2}$$

This clearly must work for some a. We could solve for it in this equation if we knew the value of $f(x_0)$. Our claim is that the function's acceleration equals a at some $x = \lambda$.

Suppose, on the contrary, that it never took on this value. Since the function is continuous, then by Lemma 1 the function's acceleration either must always be less than a or greater than a. In either case, the function having an acceleration that is always greater or less than a would mean that the correction with acceleration a would either undershoot or overshoot $f(x_0)$, respectively. This is clearly a contradiction, so the function must have the acceleration somewhere. We will generalize this idea.

3 Proof

For higher derivatives, our correction will be

$$R_n(x) = \frac{r(x - c)^{n+1}}{(n+1)!} \tag{1}$$

where r ("rate") is the necessary $(n+1)$th derivative. So that

$$f(x_0) = P_n(x_0) + R_n(x_0) \tag{2}$$

where f is continuous in the interval $[c, x_0]$.

Suppose $f^{(n+1)}$ never takes on the value r between c and x_0. By Lemma 1, either $f^{(n+1)}(x) < r$ in this interval or $f^{(n+1)}(x) > r$. We will prove that the first case yields a contradiction. The proof of the second case is identical, but with the less-than sign changed to a greater-than sign.

Lemma 2. *If $f(x) < g(x) \quad \forall x$ in $[a, b]$ then $\int_a^b f(x)\,\mathrm{d}x < \int_a^b g(x)\,\mathrm{d}x$. Same for $f(x) > g(x)$. The proof of this is left as an exercise to the reader.*

$$f^{(n+1)}(x) < r$$

By Lemma 2,

$$\begin{aligned}
\int_c^x f^{(n+1)}(x)\,\mathrm{d}x &< \int_c^x r\,\mathrm{d}x \\
f^{(n)}(x) - f^{(n)}(c) &< r(x-c) \\
\int_c^x f^{(n)}(x) - f^{(n)}(c)\,\mathrm{d}x &< \int_c^x r(x-c)\,\mathrm{d}x \\
f^{(n-1)}(x) - f^{(n-1)}(c) - f^{(n)}(c)(x-c) &< \frac{r(x-c)^2}{2} \\
\text{Integrating } (n-2) \text{ more times} &\vdots \\
f'(x) - \sum_{k=1}^{n} \frac{f^{(k)}(c)(x-c)^{k-1}}{(k-1)!} &< \frac{r(x-c)^n}{n!} \\
\int_c^{x_0} f'(x) - \sum_{k=1}^{n} \frac{f^{(k)}(c)(x-c)^{k-1}}{(k-1)!}\,\mathrm{d}x &< \int_c^{x_0} \frac{r(x-c)^n}{n!}\,\mathrm{d}x \\
f(x_0) - \sum_{k=0}^{n} \frac{f^{(k)}(c)(x_0-c)^k}{k!} &< \frac{r(x_0-c)^{n+1}}{(n+1)!} \\
f(x_0) - P_n(x_0) &< R_n(x_0) \\
f(x_0) &< P_n(x_0) + R_n(x_0)
\end{aligned}$$

This contradicts (2). Once the greater-than case is also ruled out, then by Lemma 1,

$$f^{(n+1)}(\lambda) = r \text{ for some } \lambda \text{ in } [c, x_0]$$

and by (1),

$$R_n(x_0) = \frac{f^{(n+1)}(\lambda)(x_0-c)^{n+1}}{(n+1)!} \text{ for some } \lambda \text{ in } [c, x_0] \quad \square$$