The Self-Directed Guide to Mathematics

A Handbook for Independent Discovery

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Version 1.1

Contents

1 Introduction

This book presents an alternative way of learning mathematics for highly motivated students. The best way to understand a mathematical concept is to independently discover and/or prove it yourself, without any outside help or answers. The problem with this method of learning is knowing what concepts you must derive and which results you need to arrive at. Discoveries are almost always made by accident in the context of studying a certain problem, such as compound interest, the laws governing the motion of objects, or the laws governing heat propagation. By providing these 'fertile' problems, it is hoped that a motivated reader can make such discoveries independently.

Nearly all current educational methods provide answers, solutions, and tutorials instead of questions, encouraging dependency on external help. Instead, this book provides just questions, no answers. The intended way to use this book is to discover the solutions, formulas, and proofs on your own, without consulting textbooks or internet resources.

This will result in developing much stronger mathematical skills and perhaps most importantly, confidence in your abilities. Learning this way will be neither easy nor comfortable, but it will be very rewarding. The biggest skill this book will provide, applicable even outside of mathematics, is a tolerance towards the discomfort of struggling with and thinking hard about a problem, much like an athlete develops a tolerance towards the discomfort of running long distances or swimming in cold water.

1.1 How to use this book

The best way to use this book, for middle and high school students, is to work on problems that are ahead of what you are currently learning in class. For instance, if you are currently taking Algebra 2 or Pre-calculus, work on the calculus sections. This will ensure that the solutions to the problems are not revealed in class. This book isn't a complete replacement for a standard math curriculum. For brevity, the book will deal with the most important and foundational concepts. Anything not covered or any concepts you were not able to arrive at independently can be filled in with a traditional curriculum afterwards.

Because there is a certain amount of mathematics that just needs to be taught, and because it takes a certain amount of practice before you'll be able to figure things out independently, the early sections of this book on algebra will have a greater amount of hand-holding, tips, and traditional instruction. Occassionaly, there will be some things that just need to be demonstrated and cannot be expected to be discovered independently. And by necessity, definitions will be provided throughout the book.

The background context for a problem, required definitions, and notation will be described in yellow boxes, followed by a numbered set of questions and problems. Problems with an asterisk (*) next to them are optional problems which are not needed to finish the book. The other problems will need to eventually be finished in order to complete later problems or to get a sufficient understanding of the topic. However, there is no need to complete problems in order, especially when you are stuck, since the solution to some problems will only be required much later.

This book cannot be completed in one session or in a few days. It will take many years to finish this book. There are a few problems you may never get around to solving. This is okay, because what is important is the process of independent discovery itself. Some problems will take a minute or two to solve. Other problems will take two weeks to solve, if not more. The important thing is to not give in to the urge to look up the answer, even though it will be extremely tempting.

1.2 Problem solving strategies

Here are some tips if you are stuck on a problem:

- Work on a related problem.
- Work on an easier and more basic version of the problem first.
- Make sure your problem and goal is clearly defined. This is often half of the challenge.
- For some problems, try writing out what would have to be true about your solution if you had one, and then deduce the solution from that.
- Put the problem away for awhile and work on other problems, such as the optional problems. You might need to practice your problem-solving skills for awhile before you can tackle a challenging problem. It also helps to come back to a problem with a fresh mind and fresh ideas. Sometimes, when solving something else, you find something that helps you solve a seemingly unrelated problem. Small victories on smaller problems are also important in order to stay motivated.

1.3 Other tips

I recommend keeping a dedicated math notebook. Over time, you'll go through many notebooks so you'll want to number them and try to add dates in order to keep them organized. If you have the discipline to maintain one, a table of contents is useful, so you can go back and reference your previous work. You can also maintain a central collection of important results and formulas that you use frequently. The notebook is a great thing to take with you if you need something to do when you're bored, such as a long car ride or a long wait for an appointment. Rather than taking out your cellphone and killing time, you can make some progress on difficult problems.

One tip for mathematics is to use a zig-zag order of pages to make it easier to copy things from one page to the next. Instead of going in the usual order of say pages 6,7,8,9,10,11 (left page, right page, flip page, left page, right page, flip page etc.), go in the order 7,6, 9,8, 11,10. (right page, left page, flip page, right page, left page, flip page, etc.). This allows you to copy what you've written from the left page of the previous two pages to the right page of the current two pages, so that you don't have to flip a single sheet of paper over and over as you copy a long expression.

Another fun exercise you can do is invent your own notation to make things clearer if you don't like the standard notation.

2 Basics

This section is a reference for some foundational concepts and notation used in mathematics. You do not need to know all of it right away, but you may come back to it if you do not recognize a piece of notation of terminology.

2.1 Conventions for variables

- Certain letters in mathematics tend to have certain meanings. These are not hard rules, just conventions. Do not worry if some of these words do not make sense yet.
- a, b, c, d usually denote arbitrary constants
- f, g, h usually denote functions.
- p, q, r are also frequently used for functions, but also for polynomials specifically.
- i, j, k, l, m, n typically denote integers. p and q also sometimes denote integers.
- t, w, x, y, z typically denote variables. t in particular often denotes time or a parametric parameter.
- z often denotes a complex variable
- $\vec{a}, \vec{b}, \vec{v}, \vec{w}, \vec{x}, \vec{y}, \vec{r}, \vec{s}$ are frequently used to denote vectors.
- $\alpha, \beta, \gamma, \theta$ are often used for angles.
- Capital letters often denote matrices.

2.2 Greek Alphabet Reference

- Greek letters which look the same as letters from the English alphabet are rarely used.

Lowercase	Uppercase	Name
α	A	Alpha
β	B	Beta
γ	Γ	Gamma
δ	Δ	Delta
ϵ	E	Epsilon
ζ	Z	Zeta
η	H	Eta
θ	Θ	Theta
ι	I	Iota
κ	K	Kappa
λ	Λ	Lambda
μ	M	Mu

Lowercase	Uppercase	Name
ν	N	Nu
ξ	Ξ	Xi
o	O	Omicron
π	Π	Pi
ρ	P	Rho
σ	Σ	Sigma
τ	T	Tau
υ	Υ	Upsilon
ϕ	Φ	Phi
χ	X	Chi
ψ	Ψ	Psi
ω	Ω	Omega

2.3 Basics of Sets

- A set is an unordered collection of mathematical objects, such as numbers. They are denoted using curly brackets. For example, $A = \{1, 8, 2\}, B = \{1, 2, \ldots, 999, 1000\}$
- A set can also be specified using "set-builder notation", in which a mathematical specification is given which the members must adhere to, rather than listing out the members. For example,

  $$A = \{x \mid x \text{ is a real number and } x \neq 0\}$$

  means all non-zero real numbers. The bar means "such that". So the notation reads "The set of all x such that x is a real number and x is not equal to 0".

2.4 Set operations

- $x \in S$ means that x is a member of the set S. For example, $5 \in \{1, 4, 5, 6\}$ but $3 \notin \{1, 4, 5, 6\}$
- $A \subseteq B$ means A is a subset of B. For example, $\{1, 5, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$. Note that every set is a subset of itself.
- $A \cup B$ is the "union of A and B", the set which combines all the elements from A and B. For example, $\{1, 4, 5, 8, 9\} = \{1, 5, 8\} \cup \{4, 9\}$
- There are many more set operations, but these ones are essential.

2.5 Standard Sets

- $\mathbb{N}$ - The set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Some authors exclude 0 from the natural numbers. In this book, 0 will be included.
- $\mathbb{Z}$ - The set of integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{R}$ - The set of real numbers. Examples: 5, 0, -1.2, $1.33333\ldots$, $3.14159\ldots$, $\sqrt{2}$
- $\mathbb{Q}$ - The set of rational numbers. These are all numbers of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. Examples: $\frac{1}{2}, 0, -\frac{3}{4}, \frac{4346534}{183251}$
- $\mathbb{C}$ - The set of complex numbers. These are numbers of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. These numbers will be covered later.

3 Algebra

3.1 Solving equations

1. Consider the equation $xyz = 0$. What do you know about x, y, or z?

2. Given the equation $(x-2)(x+4)=0$, what are the possible values for x?

To factor quadratics such as $x^2+7x-30$, we should start with what the final form would look like and then work backwards. Once it is factored, we will have $(x+p_1)(x+p_2)$ where p_1 and p_2 are some values we must figure out, which expands to $x^2+(p_1+p_2)x+p_1p_2$. So we know that the two values must sum to 7 (the coefficient on the x term) and multiply to -30 (the constant). Find two integers that accomplish this by checking all possible combinations of two integers, positive and negative, that multiply to -30. That is, find two integers p_1 and p_2 such that $p_1p_2=-30$ and $p_1+p_2=7$.

3. Find both solutions to the equation $x^2-2x-3=0$ by factoring it first.

The values of x of a quadratic such that $y=0$ (that is, values of x such that $ax^2+bx+c=0$) are called the *roots* of the quadratic. In other words, the roots are the x-values where the graph of the quadratic intersects the x-axis.

4. A quadratic has roots 3 and -5. The leading coefficient (of the x^2 term) is 1. Find the quadratic.

5. In general, if a quadratic has roots r_1 and r_2 and a leading coefficient of 1, what is the quadratic?

When solving the system of equations,

$$2x+2y=15-y \tag{1}$$

$$x-2y=3+y \tag{2}$$

the straightforward method is to solve for x in terms of y in the equation (2), and then substitute this expression for x into equation (1):

$$x=3+3y$$

$$\begin{gathered}2(3+3y)+2y=15-y\\6+6y+2y=15-y\\9y=9\\y=1\\x=3+3(1)=6\end{gathered}$$

An alternative method is called the *elimination method.* In the equation

$$x - 2y = 3 + y$$

note that $x - 2y$ is the *same number* as $3 + y$. A basic rule of algebra is that we can add the same number to both sides of an equation. But this number can be in different forms, as long as it's the same, so we can add $x - 2y$ to the left side of equation (1), and add $3 + y$ to the right side. The terms with y will then cancel:

$$(x - 2y) + 2x + 2y = 15 - y + (3 + y)$$

$$3x = 18$$

$$x = 6$$

Elimination cannot just be used straight away in most cases. Usually terms must be added to the equations or the entire equations multiplied by constants before elimination can be performed. This system was cherry-picked so that elimination would work right away. Whether it is faster to use elimination or substitution depends on the particular system.

Another useful fact along the same lines is that we can substitute an expression for an entire other expression, rather than only substituting an expression for a variable. For example, if

$$x + 1 + 2y = y - 5$$

$$x + 1 = 3y + 3$$

We can replace $x + 1$ in the top equation with $3y + 3$:

$$(3y + 3) + 2y = y - 5$$

$$5y + 3 = y - 5$$

$$4y = -8$$

$$y = -2$$

3.2 Functions

A function is a procedure which takes a number as input and gives an output. The input goes in parentheses after the name of the function. For example, if the function is named f, then

$$f(4) = 7$$

It is important to not confuse this with multiplication of f by 4. The input is 4 and the output is 7. We can define a function in the following way:

$$f(x) = 2x - 1$$

When defining functions, the variable x does not have a particular value here, unlike when solving algebraic equations. Instead, x is a placeholer which doesn't have any meaning. This entire expression is true for any number which is substituted in for x, a fact which means that the function gets uniquely defined:

$$f(1) = 2(1) - 1 = 1$$

$$f(-4) = 2(-4) - 1 = -9$$

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 1 = 0$$

So the full expression is really

$$f(x) = 2x - 1 \text{ for all } x$$

but "for all" is usually omitted for brevity. We could use any variable to define the function; the particular choice of x is meaningless:

$$f(x) = 2x - 1$$

$$f(y) = 2y - 1$$

$$f(☺) = 2☺ - 1$$

All of these equations define the same function.

Note that f is a function and $f(x)$ is the *value* of the function evaluated at x. However, it is common to make such statements as "the function $f(x)$". These statements are strictly incorrect, because the expression $f(x)$ is not a function. Nonetheless, this book will sometimes refer to the function $x^2 + x$ or the function $f(x + 3)$. These should be understood to be shorthand for something like "the function g such that $g(x) = x^2 + x$".

Functions can also be chained one after the other:

$$f(x) = g(h(x))$$

This is called composition.

1. If

$$g(x) = 2x^2 - 1$$
$$h(x) = x^2 + 1$$
$$f(x) = g(h(x))$$

find $f(x)$. Test a few numbers to make sure you did it correctly.

The inverse of f is denoted f^{-1}. (This is not the same of $\frac{1}{f}$. The reasoning behind this notation will be explained shortly). The inverse function is the reverse of f: it tells you which number you would need to input into f in order to get the given number as output. For example, if

$$f(5) = 2$$

then

$$f^{-1}(2) = 5$$

More specifically,

$$f^{-1}(f(x)) = x$$

2. What is $f(f^{-1}(x))$?

To find the inverse, we can write an algebraic equation of the form

$$y = f(x)$$

(Note that the variable x is a regular free variable as used in algebra, rather than a placeholder. We do *not* have "for all" after this equation). Then we can perform algebraic manipulations to get an equation of the form

$$g(y) = x$$

in which case g is f^{-1} for the following reason:

$$y = f(x)$$
$$f^{-1}(y) = f^{-1}(f(x))$$
$$f^{-1}(y) = x$$

For example, if we want the inverse of $f(x) = 2x - 1$,

$$y = 2x - 1$$

$$y + 1 = 2x$$

$$\frac{y+1}{2} = x$$

So $f^{-1}(y) = \frac{y+1}{2}$ or equivalently, $f^{-1}(x) = \frac{x+1}{2}$, or equivalently $f^{-1}(☺) = \frac{☺+1}{2}$. The variables x, y, ☺ in these last three equations are different types of variables than the ones we used while finding the inverse; we are just reusing the names. They are placeholder variables again, also called bound or dummy variables.

We may treat functions as algebraic objects in their own right:

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$f^2(x) = f(x)^2$$

(It's important to note that (x) is *not* being distributed here, because this is *not* multiplication.)

Composition is denoted $f \circ g$, so $(f \circ g)(x) = f(g(x))$. A while ago, someone noticed that this looks like multiplication and reasoned that $f \circ f \circ f$ could be written as f^3 (the compositional notation for superscripts). In other words, $f(f(f(x))) = f^3(x)$. This has the property that $f^n \circ f^m = f^{n+m}$. Most importantly, we can extend this to negative numbers and find that $f^{-1} \circ f^n = f^{n-1}$. This is where the notation for the inverse function comes from.

Notice that this conflicts with the notation established a few lines above which regards $f^3(x)$ as $f(x)^3$. The compositional notation for superscripts is rarely used; in this book, only the inverse function notation it provides will be used. Otherwise, for positive n, $f^n(x)$ will be taken to mean $f(x)^n$ from now on.

Not all functions have inverses. For example, the function $f(x) = x^2$ does not have an inverse. We have $f(2) = 4$ and $f(-2) = 4$. So $f^{-1}(4)$ could be either 2 or -2, but functions may only have a single output, not two or more. In situations like these, we separate the function into different "branches" and have an inverse for each branch. The right side of x^2 is one branch, and the left side is another. Each of these branches is invertible. $\sqrt{x}$ is the inverse of the right branch, and is thus the principle square root.

Instead of the function $f(x) = x^2$ and its inverse function, we can consider the *relation* $y = x^2$. The *inverse relation* would result if we were to swap x and y: $x = y^2$. When graphed, the inverse relation will be a sideways parabola, though it will not be the graph of a single function since each positive value of x will have two values for y on the graph.

This type of equation is called an *implicit equation*, because y is not isolated on one side and written in terms of x. The value for y given a value of x is *implicit*, since the corresponding values for y must be solved for algebraically once a value of x is plugged in. Examples of implicit equations: $y^3 + x^3 = 1$, $yx^2 + xy^2 = 3$, $\frac{y+x}{x^2+y} = 3$, $3x + 4y = 7$

3.3 Graphing functions

When asked to graph a function, you would write $y = f(x)$ and graph it as usual.

1. Graph $f(x) = 2x^2 + 5$. Then graph $g(x) = f(x+4)$.
2. In general, what happens to the graph of the function $f(x+a)$? What about $f(x-a)$? Think about why this is.
3. Given $f(x) = 2x^2 + 5$, graph $g(x) = f(2x)$. Then graph $g(x) = f\left(\frac{1}{2}x\right)$.
4. In general, what happens to the graph of the function $f(ax)$?
5. In general, what happens to the graph of the function $a + f(x)$?
6. In general, what happens to the graph of the function $af(x)$?
7. What happens to the graph of the function $f(2x+1)$? Be careful, and check your result. How about $f(ax+b)$ generally?
8. These operations are called function transformations. Find a way to write $ax^2 + bx + c$ as transformations of the function $f(x) = x^2$.
9. Find a formula which gives the solution for x in the equation $ax^2 + bx + c = 0$.
10. For each function, create a graph of the function superimposed with the graph of its inverse(s):
$$f(x) = 2x + 1$$
$$f(x) = x^2$$
$$f(x) = x^3$$
$$f(x) = \frac{x^2+1}{x}$$
$$f(x) = \frac{x^2-1}{x}$$
What relation do you notice between a function and its inverse? Given the graph of a function, how could you draw the graph of the inverse function? Think about why this is.

3.4 Parametric equations

In a parametric equation, a graph is specified not by directly relating x and y, but by introducing a third parameter which takes on values in a certain range and specifying x and y in terms of that parameter. For instance, the parameter might be time, or it might be the distance along the curve, or some other arbitrary measure of how far a point is along the curve. Parametric equations take the form

$$x = f(t)$$

$$y = g(t)$$

where $a \leq t \leq b$. That is, t varies within some range and the curve terminates at $t = a$ and $t = b$. A parametric equation need not always have a restricted range for t.

1. Given the parametric equation

$$x = 2t + 1$$

$$y = t^2 + 1$$

   Write y in terms of x

3.5 Combinatorics

The factorial of n is $n! = n(n-1)(n-2)\ldots(3)(2)(1)$. For instance, $5! = (5)(4)(3)(2)(1) = 120$

1. Find 0!

2. Can we have factorials of negative values?

3.6 Difference operator

1. Find an expression for the sum $1 + 2 + 3 + 4 + \ldots + n$

2. Consider the series x^2:

$$1, 4, 9, 16, 25, 36$$

   Find a formula for the difference between these terms. Do the same, without expansion, for x^3, and for x^n in general.

3. How about if we take every other term? Or if we take the difference between terms where the function is evaluated at intervals of 0.5? Or generally, if it is evaluated with a difference between gaps of g?

4. How about the difference of $4x^2 + 5x + 3$?

5. How about the difference of $4x^2 + 5x + 8$?

6. Find a function whose difference between terms is given by $7x^2 + 8x + 2$. Devise a procedure that will allow you to do this for any polynomial, for any gap g, without any lookup tables. This shall be called the antidifference.

7. Graph the function x^2 and the function corresponding to the differences for gaps of 0.5, 2 and 3

8. Graph the function $\frac{x^3}{6}$ along with differences for gaps of 0.5 and 2

9. Find a simplified expression for the differences of b^x

10. Find a simplified expression for the antidifference of b^x

11. Find a simplified expression for the nth antidifference of 1

3.7 Recurrence relations

A recurrence relation can be used to define a sequence of numbers in terms of previous members of that sequence. For instance, the nth term in the sequence $0, 2, 4, 6, 8, \ldots$ can be defined as

$$a_n = a_{n-1} + 2, \qquad a_1 = 0$$

The fibonacci sequence is given by

$$F_n = F_{n-1} + F_{n-2}, \qquad F_0 = 0, F_1 = 1$$

1. Find the first 6 terms of the fibonacci sequence.

A recurrence relation can also specify a 2-dimensional sequence of numbers. For example,

$$a_{n,k} = a_{n-1,k-1} + n + k \qquad a_{0,k} = k, a_{n,0} = n$$

2. Find an expression for $a_{n,k}$

3. Find an expression for the recurrence relations

$$b_{n,k} = b_{n-1,k-1} + b_{n-1,k} \qquad b_{n,0} = b_{n,n} = 1$$

$$a_{n+2} - 2a_{n+1} + a_n = n \qquad a_0 = 0$$

3.8 Binomial expansion

1. Find a pattern for the expansion of $(x + a)^n$
2. Find a formula for this expansion
3. Extend your formula to negative and real values of n

*4. Extend the formula to work for $(x_1 + x_2 + \ldots + x_m)^n$

4 Geometry

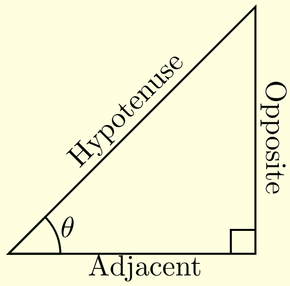
4.1 Trigonometric functions

The trigonometric functions relate the sides of a right triangle to its angles.

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

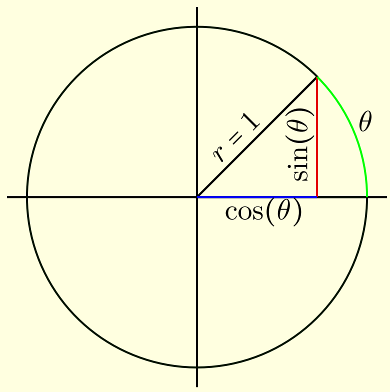
$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}$$



These relations can be remembered with the mnemonic "SOH-CAH-TOA" (sin-opposite-hypotenuse, cos-adjacent-hypotenuse, tan-opposite-adjacent).

1. Write $\tan(\theta)$ in terms of the other two trigonometric functions
2. Find the other angle of the triangle and write $\cos(\theta)$ in terms of sin. Then write $\sin(\theta)$ in terms of cos

The functions sin and cos can also be interpreted as coordinates on the *unit circle*, the circle with radius 1 centered at the origin. When working with triangles, θ is often measured in degrees. However, when working with circles, it is often measured in units of *radians* instead. An angle corresponds to a distance along an arc, as shown in green in the figure. The radian measures this arc length as multiples of the radius of the circle, hence the name "radian".



In other words, the radius of the circle is used as the unit of distance along the arc. For the unit circle, radians are the same as the arc length. Because $C = 2\pi r$, radians are usually expressed as fractions of π. For instance, $\frac{\pi}{2}$ corresponds to 90°. In general, if angular units are not specified, sin and cos use radians by default, and they will be defined as using radians for the remainder of this book. You will see later why radians are a natural unit.

As can be seen in the figure, it is standard practice for $\theta = 0$ to start on the positive x-axis, and for increases in the angle to go counter-clockwise around the unit circle.

Angles greater than 2π work as expected: they simply wrap back around. Negative angles simply go in reverse. As such, sin and cos are defined for all real numbers.

3. How can you convert between degrees and radians?

4. Relate $\sin(\theta)$ and $\cos(\theta)$, without changing the argument θ.

5. Graph $\sin(x)$, $\cos(x)$, and $\tan(x)$ separately. Make sure you can identify the x and y coordinates of the zeroes and the extrema.

6. Find a formula for $\sin(a+b)$ and $\cos(a+b)$.

7. Find a formula for $\sin(2x)$ and $\cos(2x)$.

8. Find a formula for $\sin(3x)$.

9. Rewrite the following so it does not have any multiplication of trigonometric functions:
$$\sin(a)\sin(b)$$
$$\sin(a)\cos(b)$$
$$\cos(a)\cos(b)$$

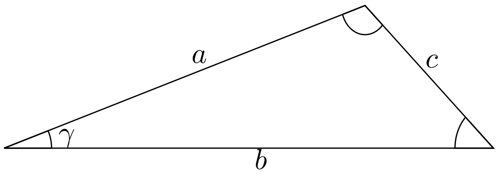
10. Find a formula for $\sin(\frac{x}{2})$ and $\cos(\frac{x}{2})$.

11. Find a formula for $\sin(\frac{x}{4})$ and $\cos(\frac{x}{4})$. Assume $0 \le x \le \frac{\pi}{2}$

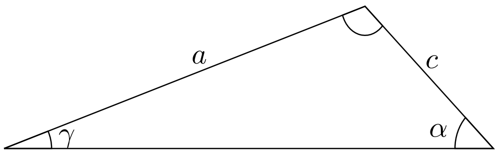
12. Write $\cos^2(x)$ and $\sin^2(x)$ in terms of trigonometric functions which are not squared.

13. Rewrite $a\sin(x) + b\cos(x)$ as a single trigonometric function.

14. Find side c of the triangle in terms of a, b and γ.



15. Relate a, c, α, and γ.



4.2 Inverse trigonometric functions

Since the trigonometric functions are not injective (infinitely many values of x give the same value of y), the inverse functions must have a restricted range. The inverse trigonometric function can be indicated either with typical inverse function notation, such as $\sin^{-1}(x)$, or by putting "arc" in front of its name, such as $\arcsin(x)$. The standard is to restrict the range of the inverse functions as follows:

$$-\frac{\pi}{2} \leq \arcsin(x) \leq \frac{\pi}{2}$$

$$0 \leq \arccos(x) \leq \pi$$

$$-\frac{\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}$$

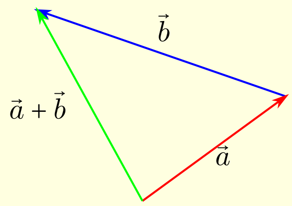
1. Graph arcsin, arccos, and arctan.
2. Relate arcsin and arccos without changing the argument
3. Relate arcsin and arccos by changing the argument
4. Find $\sin(\arccos(x))$.
5. Find $\sin(\arctan(x))$.
6. Find $\cos(\arctan(x))$.
7. Find $\sin(2\arctan(x))$.
8. Find $\cos(2\arctan(x))$.
9. Find $\tan(2\arctan(x))$.
10. Investigate $\arctan(\frac{1}{x})$

5 Linear Algebra

5.1 Vectors

A vector is a geometric object with a magnitude (length) and direction. It may also be thought of, equivalently, as a point in space. It is typically represented as a list of numbers. This list specifies the coordinates of the vector relative to the origin. All three interpretations (magnitude & direction, point in space, list of numbers) are useful depending on the problem. Vectors are denoted by placing an arrow above a variable, e.g. $\vec{v} = (5, 4, 2)$. When handwriting vectors, you may omit the bottom portion of the tip of the arrow for speed.

Vectors can be added or subtracted graphically. In this case, the beginning of vector $\vec{b}$ is placed at the tip of $\vec{a}$ rather than at the origin in order to perform the addition.



A *unit vector*, denoted with a hat above the letter, such as $\hat{u}$, is a vector with magnitude equal to 1. That is, $\|\hat{u}\| = 1$. It is also known as a direction vector.

1. Given the graphical process for vector addition, how can you do vector subtraction?

2. Determine how to add and subract vectors, based on their components, without drawing them (i.e., non-graphically).

Note: Vectors do not have an inherent location in space from which they point. They simply specify which direction to move, and how far to move in that direction. The expression $\vec{a} + \vec{b}$ starts at the origin, moves in the direction indicated by a, and then moves in the direction indicated by b.

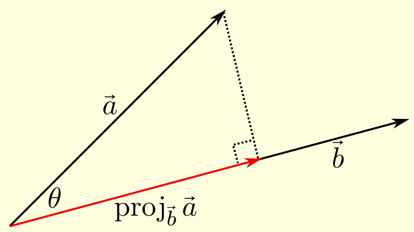
The norm of a vector, $\|\vec{v}\|$, also called the magnitude, gives the length of the vector, which is a non-negative number.

3. Find out how to calculate $\|\vec{v}\|$ based on the components of $\vec{v}$

In the terminology of vectors, real numbers which are not vectors are called "scalars." This is because a vector can be multiplied by a scalar to "scale" the vector. For instance, $2\vec{x}$ would be twice as long as $\vec{x}$ but point in the same direction. 2 would be a scalar.

4. Determine how to multiply a vector, $\vec{x}$, by a scalar, c, based on the components of $\vec{x}$. That is, find $c\vec{x}$.

The projection of vector $\vec{a}$ onto $\vec{b}$, denoted $\text{proj}_{\vec{b}}\,\vec{a}$, gives the portion of $\vec{a}$ which points in the direction of $\vec{b}$, as seen in the figure. The projection returns a vector, not a scalar.



5. Given θ as the angle between the vectors, find an expression for the magnitude of the projection, $\|\text{proj}_{\vec{b}}\,\vec{a}\|$.

We desire an operation similar to the projection, but which can be algebraically manipulated more easily. The dot product, $\vec{a}\cdot\vec{b}$, equals the magnitude of the projection of $\vec{a}$ onto $\vec{b}$ times the magnitude of $\vec{b}$. Formally, $\vec{a}\cdot\vec{b} = \|\text{proj}_{\vec{b}}\,\vec{a}\|\|\vec{b}\|$ (so it is a scalar). This means that the dot product is proportional to both $\|\vec{a}\|$ and $\|\vec{b}\|$, and that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

6. Given θ as the angle between the vectors, find an expression for $\vec{a}\cdot\vec{b}$.

7. Find out how to calculate the dot product of two vectors based on their components, without needing to know the angle between them.

8. Find an expression for $\text{proj}_{\vec{b}}\,\vec{a}$ using the dot product.

9. What is another way to write $\vec{a}\cdot\vec{a}$?

5.2 Matrices

Just as vectors can be thought of as 1-dimensional arrays of numbers, matrices are 2-dimensional arrays of numbers:

$$A_{3\times 2} = \begin{bmatrix} 5 & 10 \\ 4 & 7 \\ 3 & 2 \end{bmatrix}$$

The subscript 3×2 specifies the dimensions of the matrix, as #rows × #columns. A matrix with the same number of rows and columns is called a square matrix. Matrices are commonly represented by capital letters, while elements of the matrix are represented as lowercase letters with subscripts specifying the row and column of the element, starting at row 1 and column 1. For instance,

$$a_{32} = 2$$

Or if there are more than 9 rows,

$$b_{19,42}$$

Alternatively, the elements of a matrix expression are usually represented as follows,

$$(A + B)_{ij}$$

Vectors can be represented as matrices, either as row vectors, $\vec{v} = \begin{bmatrix} 4 & 3 & 9 \end{bmatrix}$, or as column vectors,

$$\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$$

In linear algebra, column vectors are more common.

We can also conceptualize matrices as composed of either row vectors or column vectors. While notations for specifying the rows and columns vary, this book will use the convention that the i-th row (a row vector) is given by $\vec{a_i}$, and the j-th column is given either by $\vec{A_j}$ or $\vec{a_{*j}}$. So

$$A = \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{A_1} & \vec{A_2} & \cdots & \vec{A_n} \end{bmatrix}$$

The *diagonal* entries of a matrix are those entries a_{ij} such that $i = j$.

A common linear transformation is represented as

$$\begin{aligned} w_1 &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ w_2 &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ &\vdots \\ w_m &= a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{aligned}$$

This transforms the variables $v_1, v_2, \ldots, v_n$ to $w_1, w_2, \ldots, w_m$. It is a linear transformation for technical reasons which will be learned later. Such a linear transformation can be encoded in the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

This motivates the use of *matrix multiplication* in two ways. First, so that we may write this in the form

$$\vec{w} = A\vec{v}$$

where $\vec{w}$ and $\vec{v}$ are column vectors. And secondly, so that we may chain separate transformations one after another. Suppose

$$\vec{x} = B\vec{w}$$

Then "multiplying" both sides of the original equation by B gives

$$\vec{x} = BA\vec{v}$$

If we can master matrix multiplication, we will be able to avoid a lot of tedious algebra. In general, matrix multiplication is not commutative, so $AB = BA$ is not always true. For this reason, we distinguish between "left-multiplying" and "right-multiplying" matrices on each side of an equation. Both sides must be either left or right multiplied.

1. Matrix multiplication $C = A_{p\times q}B_{m\times n}$ is only defined when $q = m$ (The "inside" numbers must be the same). What are the dimensions of C?

2. Using two transformations with 2 variables each (2 of w, 2 of v, and 2 of x), determine how to do matrix multiplication.

3. Define matrix multiplication element-wise using sigma sum notation, for instance, $C_{ij} = \ldots$ where $C = BA$.

4. Define matrix multiplication element-wise using the dot product.

5. Associativity: Is $(AB)C = A(BC)$? Justify your answer.

6. Find

$$\begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_m} \end{bmatrix} B$$

$$A \begin{bmatrix} \vec{B_1} & \vec{B_2} & \cdots & \vec{B_n} \end{bmatrix}$$

We now desire a way to undo transformations for square matrices, $A_{n\times n}$. We would like to be able to use the inverse matrix, A^{-1} to change $\vec{w}$ back into $\vec{v}$:

$$\vec{v} = A^{-1}\vec{w}$$

7. What matrix will $A^{-1}A$ give? This matrix is called the $n \times n$ identity matrix, denoted I_n, where n is the number of rows and columns.

For this reason, the inverse matrix of A will be defined as the matrix A^{-1} such that $A^{-1}A = I_n$.

8. What matrix will AA^{-1} give?

9. Find $(AB)^{-1}$, for any matrices A and B. Prove your result.

Since vectors can be added, we may wish to investigate $A\vec{y}+B\vec{y}$. If A and B were scalars, we could factor this expression into $(A+B)\vec{y}$. We will define addition for matrices as well.

10. Find out how to calculate $A+B$.

11. Find $A(B+C)$ and $(A+B)C$.

The transpose of a matrix, denoted A^T, changes rows to columns. Rigorously, $(A^T)_{ij} = A_{ji}$. Intuitively, this flips the matrix over along its diagonal entries, like flipping over a rectangular plate by holding onto its edges.

12. Find $(AB)^T$. Prove your result.

13. Find $\left(A^T\right)^{-1}$ and $\left(A^{-1}\right)^T$. Prove your results.

14. Define the dot product of two column vectors, $\vec{v}$ and $\vec{w}$, using matrix multiplication.

*15. One may object that the matrix multiplication outlined is strangely defined, and that it would be better to use row vectors. We may try to define $A \circ B = AB^T$, or some such similar operation. Investigate the properties of this operation, such as commutativity, associativity, and invertibility to see if this is actually a feasible alternative.

5.3 Determinants

Some linear transformations cannot be undone. For instance, in the transformation

$$v_1 = w_1 + 0w_2$$

$$v_2 = 2w_1 + 0w_2$$

We cannot find what w_2 was if we are only given v_1 and v_2. The information was destroyed. Such a transformation cannot be undone, and the matrix corresponding to it is said to be non-invertible. The *determinant* allows us to test whether a *square* matrix is invertible. The determinant of a matrix, denoted $|A|$ (or $\det(A)$), is a scalar value. If the determinant of a matrix is zero, the matrix is non-invertible. Otherwise, the matrix is invertible. The determinant for a matrix can be written out in full as

$$|A| = \begin{vmatrix} 5 & 3 & 2 \\ 9 & 0 & 5 \\ 7 & 2 & 1 \end{vmatrix}$$

Notice the use of vertical bars rather than brackets.

The determinant for a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For larger matrices, the determinant can be calculated by the method of cofactors. First, the minor of entry i,j of matrix A (denoted M_{ij}) is defined as the determinant of the matrix A but with the ith row and jth column removed. For instance,

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 9 & 0 & 5 \\ 7 & 2 & 1 \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} \square & \square & \square \\ 9 & \square & 5 \\ 7 & \square & 1 \end{vmatrix} = \begin{vmatrix} 9 & 5 \\ 7 & 1 \end{vmatrix}$$

Next, the cofactor is defined as

$$C_{ij} = (-1)^{i+j} M_{ij}$$

More intuitively, the sign of the minor is changed based on a checkerboard pattern:

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

Finally, the determinant is calculated by choosing a row in the matrix, multiplying each entry in the row by its corresponding cofactor, and adding up the results. For instance

$$\begin{vmatrix} 5 & 3 & 2 \\ 9 & 0 & 5 \\ 7 & 2 & 1 \end{vmatrix} = 5(+1)\begin{vmatrix} 0 & 5 \\ 2 & 1 \end{vmatrix} + 3(-1)\begin{vmatrix} 9 & 5 \\ 7 & 1 \end{vmatrix} + 2(+1)\begin{vmatrix} 9 & 0 \\ 7 & 2 \end{vmatrix}$$

In this case, we chose the first row, but any row will give the same result. We also have $|A^T| = |A|$, so this can be done column-wise on any column as well.

1. Find $|A||B|$.
2. Find $|A^{-1}|$
3. Expand

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

5.4 Solving linear systems

A linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Can be written as the matrix equation

$$A\vec{x} = \vec{b}$$

Where A is a square matrix. This can be solved directly using *Cramer's rule*:

$$x_j = \frac{|A_j|}{|A|}$$

Where A_j is the matrix A with the jth column replaced with $\vec{b}$.

*1. Using properties of determinants, prove Cramer's rule.

6 Calculus

6.1 Limits

Money placed in a bank account, or invested in other ways, can earn *interest.* The initial sum of money which is placed in the account is called the *principal*, P. Interest is the money earned from the principal. Money can be earned weekly, monthly, quarterly, yearly, etc. The amount of money earned is a percentage of the principal. That percentage is the *interest rate*, usually specified as the yearly interest rate. For example, an account with an interest rate of 2% per year and a principal of \$1000 earns $0.02 \times 1000 = \$20$ after one year. In *simple* interest, the earnings are separate from the principal, and the same amount of money is earned each year.

1. If the yearly interest rate for an account with simple interest is 6%, what is the monthy interest rate (the rate which determines how much is earned each month)? How about the biannual (twice per year, i.e., every 6 months) rate? Specifically, though you might only receive the money each year, we can talk about the money that has been earned in the intervening time.

2. If the principal in an account with simple interest is P, the yearly interest rate (expressed as a multiplicative factor rather than a percentage for simplicity) is r, and the number of years which have passed is x, find an expression for the total amount of money in the account (the principal plus the amount of money earned) after x years.

In *compound* interest, the earnings are added to the principal (compounded) periodically, perhaps yearly, quarterly, or monthly, increasing the rate at which money is earned after each compounding. The same conventions are used to find the monthly/biannual/etc. interest rate from the yearly interest rate as are used in simple interest.

3. If the money is compounded n times every year, the principal is P, the yearly interest rate is r, and the number of years which have passed is x, find an expression for the amount of money in the account after x years.

4. Check your expression: If $n = 3$, $P = \$4629$, $r = 0.037$, and $x = 5$ years then the total amount of money should be approximately \$5563

5. Suppose two competing banks start one-upping each other not by increasing the interest rate, but by increasing the number of times the money is compounded each year. First from 2 times a year, to 3 times a year, to 1000 times a year, etc. Eventually, one bank declares that the money is compounded continuously. Suppose, unrealistically, for ease of calculation, that the yearly interest rate is 100%. Find out what happens to the expression for continuous compounding, and how much money you have after 1 year. You will find a special number. Find its first 3 digits.

6. Assign your number a letter and find a general expression for continuous compound interest.

The limit is a type of operator which represents this limiting process. The expression

$$\lim_{x \to a} f(x)$$

is the value $f(x)$ "approaches" as x approaches a. We say that the limit *converges* to this value. Often, $f(a)$ is undefined, which motivates the use of the limit. In the case of

$$\lim_{x \to \infty} f(x)$$

There isn't any way to evaluate the function at infinity. $f(\infty)$ wouldn't work since ∞ is not a number.

Some limits don't exist. These limits are said to *diverge*. This means that the limit never "settles down" closer and closer to a particular value. For example,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Note that saying that the limit "equals" ∞ is just a shorthand for saying that the limit diverges to infinity. Of course, it doesn't technically equal ∞ since ∞ is not a number. Other limits, such as

$$\lim_{x \to \infty} \sin(x)$$

are said to diverge, even though they do not approach infinity, because they never settle near any particular value.

*7. Create a rigorous definition of $\lim_{x\to a} f(x)$. For example, a rigorous definition of $x > a$ might be "There exists some positive h such that $x = a + h$", where the notion of positive numbers is taken for granted.

*8. Find a way to define $\lim_{x\to\infty} f(x)$

9. The function $f(x) = 2\left(\frac{x-1}{x-1}\right)$ is equal to 2 everywhere except $x = 1$, where it is undefined since division by 0 is undefined. Is the division defined when taking the limit

$$\lim_{x\to 1} 2\left(\frac{x-1}{x-1}\right)$$

10. Use properties of limits to *formally* evaluate the following limits, or to formally find that they are divergent. If a calculator is available, verify your results numerically.

$$\lim_{x\to 2} \frac{x-2}{x^3-8}$$

$$\lim_{x\to 4} \frac{2x^2+3x+1}{5x^2+8x+9}$$

$$\lim_{x\to\infty} \frac{5x-2}{\sqrt{x-1}}$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$$

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} + \frac{x^2+3x-10}{x-2}$$

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$$

$$\lim_{x\to\infty} \frac{3x+4}{7x-4} \frac{x+3}{\sqrt[3]{x^3+2}}$$

$$\lim_{x\to 2} \frac{\sqrt{x-2}}{\sqrt{x+2}-4}$$

11. Evaluate the following limits. Prove your results in a formal way.

$$\lim_{x\to\infty} \frac{\sin(x)}{x}$$

$$\lim_{x\to\infty} \frac{\cos^2(3x)}{6x-1}$$

$$\lim_{x\to\infty} \frac{\sqrt{x}(3\sin^2(x)+4\cos(x))}{(x^2-1)(x+5)}$$

12. Evaluate the limit.

$$\lim_{x \to 5} f(f(x)) \text{ where } f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

13. Write down the properties you have found. Many of these properties have specific conditions on when it is valid to use them.

*14. Formally prove these properties using your definition of the limit

6.2 Derivatives

Let f be a function. Its derivative is denoted as f'. $f'(x)$ gives the slope of the function f at x. The derivative of the derivative (second derivative) is denoted $f''(x)$. The nth derivative is denoted $f^{(n)}(x)$

1. Find the derivative of $f(x) = x^n$
2. Let $h(x) = f(3x)$. Find the derivative of h.
3. Let $h(x) = f(g(x))$. Find the derivative of h. This is called the chain rule.
4. The product rule: Find the derivative of $h(x) = f(x)g(x)$.
5. Use a limit to create a formal definition of the derivative

*6. Prove these properties formally using the limit

7. Find the derivative of $\sin(x)$ and $\cos(x)$. Formally prove your results.
8. The quotient rule: Find the derivative of $h(x) = \frac{f(x)}{g(x)}$.
9. Investigate the derivative of b^x. Which value of b^x has the nicest derivative? This value of b will be called e. Find some of its digits.

$\log_e(x)$ will be denoted $\ln(x)$, "the natural logarithm of x".

10. Find the derivative of $\log_b(x)$
11. Given f', find the derivative of f^{-1} (the inverse function of f)
12. Find the derivative of $f(x)^{g(x)}$

13. Find the derivative of $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$

14. Find a property of limits which makes it possible to evaluate the following limits:

$$\lim_{h\to 0} \frac{\ln(1+h)}{2h+1}$$

$$\lim_{h\to 0} \frac{\ln(2+h)-\ln(2)}{\sqrt{h}}$$

$$\lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h^2}$$

$$\lim_{h\to\infty} \frac{\ln(h)}{h}$$

$$\lim_{h\to\infty} \frac{(\ln(x))^2}{e^x}$$

*15. Prove the property.

16. Explore extensions of $f^{(n)}(x)$ to real values of n, such as half derivatives

17. The partial derivative is an extension of derivatives to multivariate functions. Specifically, the prime notation $f'(x)$ only works for single variable functions. For multivariate functions, multiple notations are available for the derivative of, for example, the second argument of f evaluated at (a, b, c):

$$\partial_2 f(a, b, c)$$

$$\partial_y f(a, b, c)$$

$$D_2 f(a, b, c)$$

$$D_y f(a, b, c)$$

$$\frac{\partial f}{\partial y}(a, b, c)$$

$$f_y(a, b, c)$$

Be careful not to mix up this last notation with the component of a vector.

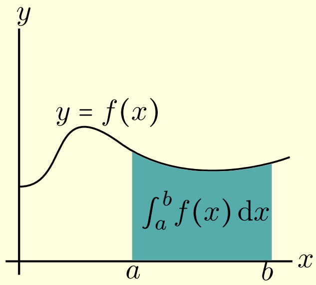
18. Find the derivative of $f(g(x), h(x))$

6.3 Antiderivatives and Integrals

1. The antiderivative: Find a function whose derivative is x^n.

2. Find the antiderivatives of $\sin(x)$ and $\cos(x)$

The integral of a function of x from a to b, denoted $\int_a^b f(x)\,\mathrm{d}x$ gives the area between the x-axis and the graph of the function between a and b. The $\mathrm{d}x$ is written to make clear which variable we are integrating with respect to. For example, $\int_0^5 x^2 y\,\mathrm{d}y$ wouldn't have a single numerical value. Instead, it is a function of x since x is a free variable.



Acceleration is the process by which an object's velocity changes. An object which falls due to gravity is accelerated as it falls, for example, because the force of gravity continues to increase its speed. On earth, gravity increases the speed of an object by $9.8\frac{\mathrm{m}}{\mathrm{s}}$ every second. This is the value of the constant g.

3. If velocity has units of $\frac{\mathrm{m}}{\mathrm{s}}$ (meters per second), what units should acceleration have?

4. Suppose a projectile is launched from the ground at an angle θ, at a speed v, and that the acceleration due to gravity is g. Find an equation for the trajectory it takes. Neglect air resistance.

5. Find $\int_0^5 x^3\,\mathrm{d}x$

6. Prove that the volume of a cone is given by $\frac{1}{3}\pi r^2 h$ where r is the radius of the base and h is the height of the cone.

7. Find
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t$$

8. Find the antiderivative of $x\sin(x)$

9. How about $x^2\sin(x)$?

10. Investigate the antiderivative of $f(x)g(x)$

11. Investigate the antiderivative of $f(g(x))$

12. Find the antiderivative of $\sin(x)\cos^2(x)$

13. Find the antiderivative of $\frac{\sqrt{x}}{1+x}$

14. Find an expression for the antiderivative of $x^n e^x$

15. Find the antiderivative of $f^{-1}(x)$
16. Find the antiderivative of $\tan(x)$
17. Find the antiderivative of $\arcsin(x)$
18. Find the antiderivative of $\ln(x)$
19. Find the antiderivative of $\cos^2(x)$
20. Find the antiderivative of $\frac{1}{\sqrt{1+x^2}}$
21. Find the $(n-1)$th antiderivative of $(x+a)$
22. Use a limit to create a definition for $\int_a^b f(x)\,\mathrm{d}x$

The integral

$$\int_a^\infty f(x)\,\mathrm{d}x$$

is not technically valid, since ∞ is not a number. But it can be defined as

$$\lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

23. Find these integrals, or find that they diverge.

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x$$

$$\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$$

$$\int_0^1 \frac{1}{x}\,\mathrm{d}x$$

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x$$

$$\int_0^1 \ln(x)\,\mathrm{d}x$$

24. Find a formula for the distance along the curve of the function x^2

*25. Find the antiderivative of $\frac{1}{\sin(x)}$

*26. Find the antiderivative of $\frac{\cos(x)}{3\sin(x)+2\tan(x)}$

*27. Find the antiderivative of $\arcsin(\frac{1}{x})$

28. Find
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(t)\,\mathrm{d}t$$

29. Find
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a(x)}^{b(x)} f(x,t)\,\mathrm{d}t$$
Simplify your result as much as possible.

30. Find
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x^2} \frac{e^{xt}-1}{t}\,\mathrm{d}t$$

6.4 Infinite sums

1. Find a formula for the sum of
$$\sum_{n=0}^{\infty} b^n$$
where $-1 < b < 1$

2. Find a way to represent functions as an infinite sum. Functions which can be written using the method you will find are called "analytic." Some functions cannot be written this way.

3. Using this method, write $\sin(x)$ as an infinite sum. Make sure you can give your formula to someone who doesn't know anything about $\sin(x)$ so that they can approximate its values.

4. Write the following as infinite sums:
$$e^x$$
$$\cos(x)$$
$$\sin(x^2)$$
$$\frac{1}{x}$$
$$\frac{1}{1-x}$$
$$x^2\sin(x)$$

*5. If n terms are used in such a sum, find a way to determine an upper-bound for the error between the sum and the true value of the function f at x, without knowing the true value of $f(x)$.

6. Prove your result.

7. Determine whether the following sums converge or diverge. Formally prove your result.

$$\sum_{n=0}^{\infty} \frac{1}{x}$$

$$\sum_{n=0}^{\infty} \frac{1}{x^2}$$

8. Compute

$$\sum_{n=1}^{\infty} \frac{-1}{n(n+1)}$$

$$\sum_{n=0}^{\infty} \ln\left(\frac{\cos\left(e^{-n^2-2n-1}\right)}{\cos\left(e^{-n^2}\right)}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(2n+1)}$$

7 Differential Equations

1. Find a non-zero function whose derivative is equal to itself: $f'(x) = f(x)$

2. Is the solution unique? If not, find every solution.

3. Can a solution have the property that $f(0) = 2$? If so, find it, and determine whether it's unique. If it's not unique, find all such solutions. This type of specification is called an "initial value problem" (IVP).

4. Find a closed-form solution to the equation $f'(x) = h(f(x))$

5. Find a closed-form solution to the equation $f'(x) = g(x)h(f(x))$

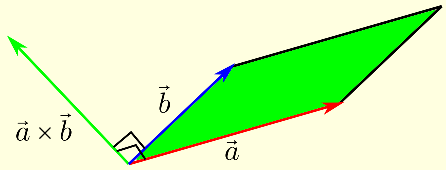
6. Find a closed-form solution to the equation $f''(x) = h(f(x))$

7. Find a closed-form solution to the equation $f'(x) = p(x)f(x) + q(x)$

8 Multivariable calculus

8.1 Cross product

The cross product of two 3-dimensional vectors, $\vec{a} \times \vec{b}$ ("a cross b") gives the *area vector* of the parallelogram formed by the two vectors. The area vector's magnitude is the area of the parallelogram, and its direction is perpendicular to the surface of the parallelogram, so that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.



A 3-dimensional coordinate system can be either right-handed (the standard) or left-handed. In a right-handed coordinate system, take your right hand, point your index finger in the direction of the x-axis, point your middle finger in the direction of the y-axis, and point your thumb upwards, perpendicular to your index finger and middle finger, as though you were forming a finger gun. Your thumb will point in the direction of the z-axis. This is called a right-hand rule. For a left-handed coordinate system, the same thing holds but with the left hand. In a right-handed coordinate system, the cross product also satisfies a right hand rule. In the equation

$$\vec{a} \times \vec{b} = \vec{c}$$

The direction of $\vec{c}$ is given by your thumb when you point your index finger in the direction of $\vec{a}$ and point your middle finger in the direction of $\vec{b}$.

1. If θ is the angle between two vectors, find an expression for the magnitude of their cross product.

2. Without writing anything down, find $\vec{a} \cdot (\vec{a} \times \vec{b})$.

3. What is $\vec{b} \times \vec{a}$ in terms of $\vec{a} \times \vec{b}$?

Vectors can be written as linear combinations of the *standard basis* unit vectors, with i-j-k notation:

$$\hat{\imath} = (1, 0, 0)$$

$$\hat{\jmath} = (0, 1, 0)$$

$$\hat{k} = (0, 0, 1)$$

So that $(5, 1, 7) = 5\hat{\imath} + \hat{\jmath} + 7\hat{k}$.

The cross product satisfies

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

and

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

4. Use these properties, along with i-j-k notation and the right hand rule, to write $\vec{a} \times \vec{b}$ in terms of the components of $\vec{a}$ and $\vec{b}$.

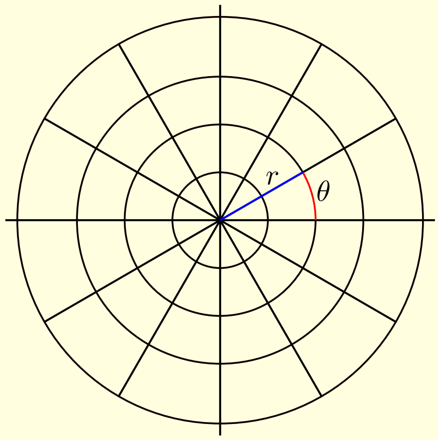
Using i-j-k notation, we can establish an easy-to-remember method for calculating the cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

8.2 Coordinate Systems

1. Define a circle in terms of a parametric equation ("parameterize" the circle).

In polar coordinates, instead of specifying points as (x, y) coordinates, they are specified as an angle on a circle with a certain radius, (r, θ). r is always positive. We can also specify functions in polar coordinates as $r = f(\theta)$ or $\theta = f(r)$



2. Find the equation for the unit circle in polar coordinates.

3. Find the x and y coordinates in terms of the polar coordinates r and θ.

Cylindrical coordinates are like polar coordinates, but 3-dimensional and specified with (r, θ, z). It is customary for the z axis to point out of the page.

Spherical coordinates are like polar coordinates, but with the third dimension specified by ϕ, the angle between the point and the z axis. ρ is sometimes used instead of r.

4. Find the x, y, and z coordinates in terms of the spherical coordinates (r, θ, ϕ).

8.3 Multiple Integration

Like a regular integral of a single variable function is the area under the curve between a set of bounds, the double integral of a function of two variables is the volume under the surface, within some specified region. Since this region is not always square, it is not always as simple as specifying two sets of bounds. For this reason, we often give a letter to specify the region (domain of integration) and write the integral as such:

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y$$

or

$$\iint_D f(x,y)\,\mathrm{d}y\mathrm{d}x$$

Where D is the domain, which can be described verbally or given a mathematical specification.

1. Evaluate
$$\iint_D 5 - x - y\,\mathrm{d}x\mathrm{d}y$$
where D is the area between the x-axis, y-axis, and the line $y = 3 - x$.

2. Change the order of integration to evaluate the following iterated integral:
$$\int_0^5 \int_x^5 e^{y^2}\,\mathrm{d}y\,\mathrm{d}x$$

3. Some problems are suited to different coordinate systems, such as polar coordinates. In general, we wish to use different variables in our integration. Find a way to evaluate
$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y$$
by changing the variables to v and w, where
$$x = g(v,w)$$
$$y = h(v,w)$$
The new region of integration will be called R. Feel free to bend the rules to arrive at your result.

4. Generalize your result for triple and higher integrals.

5. Use your result to show that the volume of a sphere is $\frac{4}{3}\pi r^3$

8.4 Vector functions

A parametric equation can be condensed into a vector function

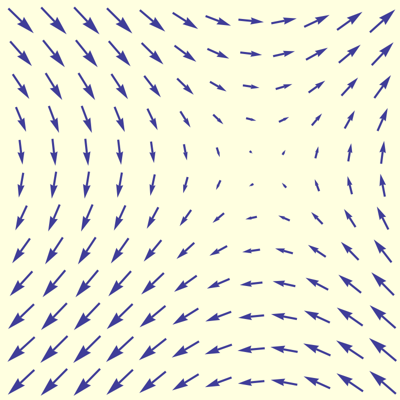
$$\vec{w} = \vec{f}(t)$$

1. Find a formula for the distance along a parametric curve.

A vector function can also take a vector as its argument:

$$\vec{w} = \vec{f}(\vec{v})$$

This is equivalent to a multivariate function which returns a vector. When its vector argument has the same dimension as its value, the function is called a *vector field*. It assigns a vector to every location in space, which allows it to be visualized. To the right is a graph of $f(x, y) = (\sin(y), \sin(x))$



Flux, a scalar, describes the amount that a 3d vector field penetrates a surface, whether that surface be a sphere, a half-sphere, a plate, etc. However, we will generalize this concept for 2d space as well, so that it works for circles, ellipses, arcs, etc. A concrete example of flux is the amount of a fluid such as air or water leaving an imaginary closed surface such as a sphere, where the velocity vector of the fluid at each point in space is described as a vector field.

A closed surface is one with no holes, i.e., no way to leave the region it encloses. An integral over a closed surface can denoted as follows:

$$\oint_S \ldots \, \mathrm{d}S$$

where $\mathrm{d}S$ is an infinitesimal element of the surface.

2. Find an integral for the flux of a 2d vector field through a closed surface, S. Parameterize the surface.

3. This integral gives a measure of how much fluid is leaving the surface. We desire a similar measure that gives some measure of how much fluid is leaving a point in space.

9 Complex Analysis

9.1 Special functions

1. Find a way to approximate $n!$ for very large values of n. Specifically, you should have that $\lim_{n\to\infty} \frac{n!}{f(n)} = 1$ where $f(n)$ is your approximation.

2. Find a way to extend $n!$ to real values of n.

When extended beyond integers, the factorial function is called the Pi function, and is denoted $\Pi(x)$. For historical reasons, this function is rarely used. Instead, the gamma function, $\Gamma(x) = \Pi(x-1)$, is more widely used. Use whichever you prefer.

9.2 Complex derivatives

A complex function can be thought of as a vector field, with the real part as the x-component and the imaginary part as the y-component.

1. Like vector fields, a derivative for a complex function can be taken in different directions. Define a directional derivative for complex integration.

2. Does the derivative of z^n depend on the direction? If so, how?

3. How about the functions $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, and $\mathrm{abs}(z)$?

9.3 Contour integration

1. We can take line integrals in the complex plane. Use theorems from vector calculus to find theorems for complex integration. Find a nice line integral theorem that doesn't need pesky Re and Im functions everywhere. Prove your theorem.

2. Find the integral of $1/z$ along a circle centered at the origin with radius 1. Now find it with arbitrary radius r. Now with the circle centered at z_0.

3. Characterize the singularities of the following functions:

$$\frac{1}{(z+2)(z-3)}$$

$$\frac{z+2}{(z^2-2z+1)^2}$$

$$\frac{e^z}{\sin(z)}$$

4. Use complex analysis to evaluate the following real integrals.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

$$\int_0^{2\pi} \frac{1}{5-4\sin x}\,\mathrm{d}x$$

$$\int_0^{2\pi} \frac{\cos x}{13+12\cos x}\,\mathrm{d}x$$

$$\int_0^{\infty} \sin(x^2)\,\mathrm{d}x$$

10 Challenge problems

These problems range from tricky but doable to incredibly difficult. Some of these are problems I have not yet been able to solve myself.

1. Find the antiderivative of $\frac{\sqrt{1+x}}{\sqrt{1-x}}$
2. Evaluate $\int_0^\infty \frac{\sin(x)^2}{x^2(x^2+1)}\,\mathrm{d}x$
3. Prove that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$
4. Find the cubic formula, which gives the three solutions to the cubic equation $ax^3+bx^2+cx+d=0$